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Some Problems in Algebraic and Extremal Graph Theory.

Edward Tauscher Dobson

Louisiana State University and Agricultural & Mechanical College

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SOME PROBLEMS IN ALGEBRAIC
AND
EXTREMAL GRAPH THEORY

A Dissertation

Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

in

The Department of Mathematics

by
Edward Tauscher Dobson
B.S., University of North Texas, 1988
M.A., University of North Texas, 1989
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ABSTRACT

In this dissertation, we consider a wide range of problems in algebraic and extremal graph theory. In extremal graph theory, we will prove that the Tree Packing Conjecture is true for all sequences of trees that are ‘almost stars’; and we prove that the Erdős-Sós conjecture is true for all graphs G with girth at least 5. We also conjecture that every graph G with minimal degree k and girth at least $2t + 1$ contains every tree T of order $kt + 1$ such that $\Delta(T) \leq k$. This conjecture is trivially true for $t = 1$. We prove the conjecture is true for $t = 2$ and that, for this value of t , the conjecture is best possible. We also provide supporting evidence for the conjecture for all other values of t .

In algebraic graph theory, we are primarily concerned with isomorphism problems for vertex-transitive graphs, and with calculating automorphism groups of vertex-transitive graphs. We extend Babai’s characterization of the Cayley Isomorphism property for Cayley hypergraphs to non-Cayley hypergraphs, and then use this characterization to solve the isomorphism problem for every vertex-transitive graph of order pq , where p and q distinct primes. We also determine the automorphism groups of metacirculant graphs of order pq that are not circulant, allowing us to determine the nonabelian groups of order pq that are Burnside groups. Additionally, we generalize a classical result of Burnside stating that every transitive group G of prime degree p , is doubly transitive or contains a normal Sylow p -subgroup to all p^k , provided that the Sylow p -subgroup of G is one of a specified family. We

believe that this result is the most significant contained in this dissertation. As a corollary of this result, one easily gives a new proof of Klin and Pöschel's result characterizing the automorphism groups of circulant graphs of order p^k , where p is an odd prime.

CHAPTER 1

INTRODUCTION

Combinatorics seems to be the branch of mathematics that most easily lends itself to posing questions which are relatively easy to understand but whose solutions are exceedingly difficult. In this dissertation, we will consider several such questions. The first is a conjecture of Erdős and Sós stating that every graph of order n and at least $\lfloor n(k-1)/2 \rfloor + 1$ edges contains every tree of order $k+1$. We will verify that this conjecture is true for every such graph that has girth at least 5. The second such question is a conjecture of Gyárfás and Lehel that states that every sequence of trees T_1, \dots, T_n , $|V(T_i)| = i$, can be packed into K_n . We will verify this conjecture for sequences of trees that are ‘almost’ stars. These conjectures, with related problems, are the problems in extremal graph theory that we will consider.

In algebraic graph theory, we consider problems of a much more fundamental nature. One of the most basic questions that one can ask in graph theory is when are two graphs isomorphic? In general, this question cannot be answered in any but the most superficial way. However, if we restrict our attention to special classes of graphs, the problem is much more attemptable. We will consider the special class of vertex-transitive graphs, and more usually, Cayley graphs. Clearly every graph has an automorphism group. Another fundamental question is given a graph, what is its automorphism group? Again, this problem in general is impossible, and we will again restrict our attention to vertex-transitive graphs, and more usually, circulant

graphs. These problems, with related questions, are the problems in algebraic graph theory that we will consider.

Throughout this dissertation, notation is relatively standard. For graph theoretic terms not defined in this dissertation, see [12], and for permutation group theoretic terms, see [58]. A *graph* Γ is an ordered pair (V, E) , where V is a set called the *vertex set* and $E \subseteq \{(x, y) : x, y \in V, x \neq y\}$ is called the *edge set*. We usually denote V by $V(\Gamma)$ and E by $E(\Gamma)$. An *isomorphism* between two graphs Γ_1 and Γ_2 is a bijection $\alpha : V(\Gamma_1) \rightarrow V(\Gamma_2)$ such that $\alpha(E(\Gamma_1)) = E(\Gamma_2)$. An *automorphism* is an isomorphism from a graph to itself. The set of all automorphism of a graph Γ form a group under composition called the *automorphism group* and is denoted $\text{Aut}(\Gamma)$.

CHAPTER 2
PROBLEMS IN ALGEBRAIC
GRAPH THEORY

In recent years, much interest has been expressed in *vertex-transitive graphs*, that is, graphs whose automorphism group act transitively on the graph's vertex set. In particular, a great deal of this interest has been generated by two conjectures, the most celebrated of the two being the following conjecture of Lovász [39] in 1970.

Conjecture 2.1. Every connected vertex-transitive graph contains a Hamiltonian path.

Although being a vertex-transitive graph is a very strong property for a graph to possess, no general results have yet been proven about Lovász's conjecture using only the property. To obtain results on the conjecture, it has been necessary to either prove that certain vertex-transitive graphs have additional properties, or to make assumptions about a given vertex-transitive graph. To illustrate this, we will prove that if p is prime then every connected vertex transitive graph of order p contains a Hamiltonian path, and, if in addition $p \neq 2$, then every such graph is Hamiltonian.

Let n be a positive integer and $S \subseteq \mathbb{Z}_n$ such that $0 \notin S$ and $-S = S$. Define a graph $\Gamma(n, S)$ by $V(\Gamma(n, S)) = \mathbb{Z}_n$ and $E(\Gamma(n, S)) = \{(i, j) : i - j \in S\}$. The graph $\Gamma(n, S)$ is said to be a *circulant graph* of order n with *connection set* S . It is easy

to verify that a graph Γ of order n is isomorphic to a circulant graph of order n if and only if there exists $\rho \in \text{Aut}(\Gamma)$ such that $\langle \rho \rangle$ is cyclic of order n . Now, let Γ be a vertex-transitive graph of order p , where p is a prime. As the order of an orbit of a permutation group divides the order of the group, $p \mid |\text{Aut}(\Gamma)|$. Hence $\text{Aut}(\Gamma)$ contains a p -cycle so that Γ is isomorphic to a circulant graph of order p . This fact was first shown by Turner [56]. It thus suffices to show that every connected circulant graph Γ of order p contains a Hamiltonian path, and if $p \geq 3$, then every such graph is Hamiltonian. For $p = 2$, this is trivial. If $p \geq 3$, let $s \in S$. Then $\langle s \rangle = \mathbb{Z}_p$ so that $0 \ s \ 2s \ \dots \ (p-1)s \ 0$ is a Hamiltonian cycle in $\Gamma(n, S)$.

Let G be a group and $S \subseteq G$ such that $1_G \notin S$ and $S^{-1} = S$. Define a graph $\Gamma(G, S)$ by $V(\Gamma) = G$ and

$$E(\Gamma(G, S)) = \{(sg, g) : s \in S, g \in G\}.$$

We say that $\Gamma(G, S)$ is a *Cayley graph* of G with *connection set* S . Let $g_L : G \rightarrow G$ by $g_L(h) = gh$. It is easy to verify that $G_L = \langle g_L : g \in G \rangle \leq \text{Aut}(\Gamma(G, S))$, and Sabidussi [53] showed that a graph Γ is a Cayley graph of G if and only if $G_L \leq \text{Aut}(\Gamma)$. We remark that the most significant result yet obtained on Conjecture 2.1 was obtained by Keating and Witte [35] who proved that if Γ is a connected Cayley graph of a group G and the commutator subgroup of G is cyclic of order p , then Γ is Hamiltonian.

Hence one method of attacking Lovász's conjecture is to characterize, in some meaningful way, all vertex-transitive graphs of a given order, and then determine

whether or not every connected graph in the characterization contains a Hamiltonian path. Until recently, for general n , this problem was hopeless. However, the classification of the finite simple groups in 1980 provides an essential tool for attempting this problem. Before discussing this further, we need some definitions.

Let G be a transitive permutation group acting on a set Ω . If there exists a set $B \subseteq \Omega$, $B \neq \emptyset$, such that $\alpha(B) = B$ or $\alpha(B) \cap B = \emptyset$ for all $\alpha \in G$, then we say that B is a *block* of G . If $B = \Omega$ or $B = \{x\}$, where $x \in \Omega$, then B is a *trivial block*. If G admits a nontrivial block, then G is said to be *imprimitive*. Otherwise, G is *primitive*.

Let m and n be positive integers and let α be an element of \mathbb{Z}_n^* , the group of units of \mathbb{Z}_n . Let $V = \{v_j^i : i \in \mathbb{Z}_m, j \in \mathbb{Z}_n\}$. Define $\rho, \tau : V \rightarrow V$ by

$$\rho(v_j^i) = v_{j+1}^i$$

and

$$\tau(v_j^i) = v_{\alpha j}^{i+1}.$$

Let $\mu = \lfloor m/2 \rfloor$ and $S_0, S_1, \dots, S_\mu \subseteq \mathbb{Z}_n$ such that

- (i) $0 \notin S_0 = -S_0$,
- (ii) $\alpha^m S_r = S_r$, $0 \leq r \leq \mu$,
- (iii) if m is even, then $\alpha^\mu S_\mu = S_\mu$.

Define a graph $\Gamma = \Gamma(m, n, \alpha, S_0, S_1, \dots, S_\mu)$ by $V(\Gamma) = V$ and

$$E(\Gamma) = \{v_j^i v_h^{i+\tau} : 0 \leq \tau \leq \mu \text{ and } h - j \in \alpha^i S_\tau\}.$$

The graph Γ is said to be an (m, n) -*metacirculant graph*. These graphs were first studied by Alspach and Parsons [3], and, among other things, they proved that Γ is an (m, n) -metacirculant graph if and only if $\langle \rho, \tau \rangle \leq \text{Aut}(\Gamma)$.

Let $p > q$ be prime. Marušič [45] proved that if Γ is a vertex-transitive graph of order pq such that the minimal transitive subgroup of $\text{Aut}(\Gamma)$ is not simple, then Γ is isomorphic to a (q, p) -metacirculant graph. We give an independent proof of this fact in Chapter 3. Let G be a transitive group acting on Ω , and let G act on $\Omega \times \Omega$ by $g(\alpha, \beta) = (g(\alpha), g(\beta))$. Let $\mathcal{O}_0, \mathcal{O}_1, \dots, \mathcal{O}_n$ be the orbits of G acting on $\Omega \times \Omega$. We call the orbit $\{(\alpha, \alpha) : \alpha \in \Omega\}$ the *trivial orbit*. Assume \mathcal{O}_0 is the trivial orbit, and define directed graphs $\Gamma_1, \dots, \Gamma_n$ by $V(\Gamma_i) = \Omega$ for all $1 \leq i \leq n$, and $E(\Gamma_i) = \mathcal{O}_i$. The graphs Γ_i are *orbital digraphs* of G . Note that $G \leq \text{Aut}(\Gamma_i)$ for all $1 \leq i \leq n$. The orbits of (α, β) and (β, α) are said to be *paired orbits*, and if they are the same, the orbit is *self paired*. Observe that if \mathcal{O}_i and \mathcal{O}_j are paired orbits, but not self paired orbits, then we may define a graph Γ by $V(\Gamma) = \Omega$ and $E(\Gamma) = \mathcal{O}_i \cup \mathcal{O}_j$. Then $G \leq \text{Aut}(\Gamma)$. We remark that if $|\Omega| = pq$ as above, then $n \geq 3$ [52]. Hence to obtain a reasonable characterization of vertex-transitive graphs of order pq , we must know all *simply primitive groups* of degree pq , that is, simple groups that have a primitive action on pq points. Further, such groups do exist, and graphs with simply primitive automorphism groups also exists. Marušič [41] also showed that the automorphism group of the line graph of the Petersen graph is

simple and imprimitive. Hence without the classification of the finite simple groups, we cannot even obtain a reasonable characterization of vertex-transitive graphs of order pq . We would like to point out that determining all primitive groups of order $2p$ was a classical problem (see [58]) that was only solved using the classification of the finite simple groups.

Characterizations of vertex-transitive graphs have been obtained for graphs of order p , p^2 [43], p^3 [43], and pq [18], [46], [52], [57], and Lovász's conjecture has been verified for graphs of order p , p^2 [43], p^3 [43], $2p^2$ [44], $4p$ [42], and for most graphs of order pq [45]. Note that the proofs of most of the preceding results also determine which graphs are actually Hamiltonian. For example, all connected vertex-transitive graphs of order p , p^2 , and p^3 are Hamiltonian.

The second conjecture that has aroused much interest is the following conjecture of Ádám [1] in 1968.

Conjecture 2.2. Let $\Gamma(n, S)$ and $\Gamma(n, S')$ be isomorphic circulant graphs. Then there exists $\alpha \in \mathbb{Z}_n^*$ such that $\alpha(S) = S'$.

Essentially, Ádám conjectured that two Cayley graphs of \mathbb{Z}_n are isomorphic if and only if they are isomorphic by a group automorphism of \mathbb{Z}_n . One can easily ask the more general question of given two Cayley graphs of G , when are they isomorphic by a group automorphism of G ? The first positive result on Adam's conjecture was for p a prime, by Turner [56]. However, the result actually follows from the following classical theorem of Burnside [15], proved in 1901.

Theorem 2.3. (Burnside's Theorem) Let p be a prime and G a transitive permutation group of degree p acting on Ω . Then G is doubly transitive or we may relabel Ω with elements of \mathbb{Z}_p so that $G \leq \{ax + b : a \in \mathbb{Z}_p^*, b \in \mathbb{Z}_p\} = N(p)$.

Note that if $\text{Aut}(\Gamma(p, S))$ is doubly transitive, then $\Gamma = K_p$ or E_p , the complete graph or the empty graph of order p , in which case the result is trivial. If not, then $\text{Aut}(\Gamma(p, S)) \leq N(p)$. Now, $N(p) = N_{S_p}[(\mathbb{Z}_p)_L]$ and $(\mathbb{Z}_p)_L$ is a Sylow p -subgroup of S_p . Let $\Gamma(p, S')$ be isomorphic to $\Gamma(p, S)$ with $\delta : \Gamma(p, S) \rightarrow \Gamma(p, S')$ an isomorphism. Then $\delta^{-1}(\mathbb{Z}_p)_L \delta = (\mathbb{Z}_p)_L$ so that $\delta \in N(p)$. As $(\mathbb{Z}_p)_L \leq \text{Aut}(\Gamma(p, S))$, $(\mathbb{Z}_p)_L \leq \text{Aut}(\Gamma(p, S'))$, we may assume that $\delta(x) = ax$, $a \in \mathbb{Z}_p^*$. It is then easy to verify that $aS = S'$. In Chapter 5 we will extend Burnside's important and powerful result to prove that if G is a transitive permutation group of degree p^k with Sylow p -subgroup Π , where Π is one of a specified family, then $\Pi \triangleleft G$ or G is doubly transitive.

Let Γ be a Cayley graph of G . We call Γ a *CI-graph of G* if whenever Γ' is a Cayley graph of G then Γ and Γ' are isomorphic if and only if they are isomorphic by a group automorphism of G . (Here CI stands for Cayley Isomorphism.) If every Cayley graph of G is a CI-graph of G , we say that G is a *CI-group with respect to graphs*. In many cases, it will be advantageous to consider the same question about other types of 'combinatorial objects'.

Let H be a set, and $E \subseteq 2^H \cup 2^{2^H} \cup \dots$. We say that the ordered pair $X = (H, E)$ is a *combinatorial object*. We call H the *vertex set* and E the *edge set*.

If $E \subseteq 2^H$, then X is a *hypergraph*. An *isomorphism* between two combinatorial objects $X = (H, E)$ and $X' = (H', E')$ is a bijection $\delta : H \rightarrow H'$ such that $\delta(E) = E'$. An *automorphism* of a combinatorial object X is an isomorphism from X to itself. Let G be a group and $X = (G, E)$ a combinatorial object. Then X is a *Cayley object of G* if and only if $G_L \leq \text{Aut}(X)$. A Cayley object X of G is a *CI-object of G* if and only if whenever X' is a Cayley object of G isomorphic to X , then an isomorphism of X and X' is given by some $\alpha \in \text{Aut}(G)$. Similarly, G is a *CI-group with respect to \mathcal{K}* if and only if every Cayley object in the class of combinatorial objects \mathcal{K} is a CI-object of G , and a *CI-group* if G is a CI-group with respect to every class \mathcal{K} of combinatorial objects. Babai [4] characterized this property in the following fashion.

Lemma 2.4. For a Cayley object X of G in \mathcal{K} the following are equivalent:

- (i) X is a CI-object of G ,
- (ii) whenever $\varphi \in S_G$ such that $\varphi^{-1}G_L\varphi \leq \text{Aut}(X)$, then $\varphi^{-1}G_L\varphi$ and G_L are conjugate in $\text{Aut}(X)$.

We observe that the isomorphism problem for the class of designs has a much older history than the corresponding question for graphs. In the early 1930's, Bays [7], [8], [9], [10], and Lambossy [40] proved Lemma 2.4 for designs, and used it to prove that Cayley designs of order \mathbb{Z}_p are CI-designs of \mathbb{Z}_p , where p is prime. Further,

Alspach and Parsons [2], proved Lemma 2.4 for graphs. Using Lemma 2.4, Palfy [49] proved the following surprising result.

Theorem 2.5. G is a CI-group with respect to every class of combinatorial objects if and only if $|G| = 4$, or $G \cong \mathbb{Z}_n$ and $(n, \varphi(n)) = 1$, where φ is Euler's phi function.

The following groups have been show to be CI-groups with respect to graphs.

- (i) \mathbb{Z}_p , p a prime (Alspach and Parsons [2], Babai [4], Djoković [20], Turner [56]),
- (ii) \mathbb{Z}_{pq} , p and q distinct primes (Alspach and Parsons, [2], Klin and Pöschel [36]),
- (iii) \mathbb{Z}_n , n square free (Gol'fand, [31]),
- (iv) $\mathbb{Z}_p \times \mathbb{Z}_p$, p a prime (Godsil [30]),
- (v) $4p$, p and odd prime (Godsil [30]),
- (vi) D_p , p a prime (Babai [4]).

where D_p is the *dihedral group* of order $2p$. Furthermore, Ádám's conjecture has been shown to be false [28], and in particular, if $p^2|n$, p an odd prime, then \mathbb{Z}_n is not a CI-group with respect to graphs [2]. In Chapter 6 we shall show that \mathbb{Z}_p^3 is a CI-group with respect to graphs, for p a prime, providing a partial solution to a conjecture of Babai and Frankl [5]. The techniques used in proving \mathbb{Z}_p^3 is a CI-group with respect to graphs are quite general, and using these techniques we have been able to find necessary and sufficient conditions to determine when two

Cayley graphs of a group G of order p^3 are isomorphic [23], [24], [25], although not all of those results will be presented in this dissertation. Together with the characterization of vertex-transitive graphs of order p^3 mentioned above, these results together provide a classification of vertex-transitive graphs of order p^3 . We note that Xu [59] independently proved that \mathbb{Z}_p^3 is a CI-graph with respect to graphs.

In Chapter 4, we shall also show that if G is a nonabelian group of order pq , $p \geq 3 \geq q \geq 2$ distinct primes, then G is a CI-group with respect to graphs, and that if $p > q > 3$, then G is not a CI-group with respect to graphs, but that if Γ is a Cayley graph of G that is not a CI-graph of G , then Γ is also isomorphic to a circulant graph of order pq . We remind the reader that Babai [4] has shown that if $p > q = 2$ then G is a CI-group with respect to graphs. These results motivate the following definition. If a group G_1 is not a CI-group with respect to some class \mathcal{K} of combinatorial objects, but every Cayley object of G_1 that is not a CI-object of G_1 with respect to \mathcal{K} is also a Cayley object of some group G_2 , and G_2 is a CI-group with respect to \mathcal{K} , we say that G_1 is a *weak CI-group via G_2 with respect to \mathcal{K}* . Hence we show that G is a weak CI-group via \mathbb{Z}_{pq} with respect to graphs. Note that if G' is a group and G' is not a CI-group with respect to graphs, then G' is not necessarily a weak CI-group via G'' for any group G'' . Klin and Pöschel [36], [37] showed that \mathbb{Z}_{p^k} is not a weak CI-group for $p \geq 2$ and $k \geq 3$ (we provide an independent proof of this fact in Chapter 5). In proving the above results, we make extensive use of Lemma 2.4.

In [3], Alspach and Parsons posed the problem of determining necessary and sufficient conditions for two (m, n) -metacirculant graphs to be isomorphic. If the metacirculant graphs are Cayley graphs, then Lemma 2.4 provides a useful tool to attack this problem. However, not all (m, n) -metacirculant graphs are Cayley graphs. For example, the Petersen graph is a metacirculant graph but is not a Cayley graph (see Example 4.3). We will solve this problem for (q, p) -metacirculant graphs, $q < p$ primes. Before solving this problem, we first generalize Lemma 2.4 to non-Cayley graphs.

Let X and Y be vertex-transitive hypergraphs. Let

$$A = \{((x_1, y_1), (x_2, y_2), \dots, (x_r, y_r)) : (x_1, \dots, x_r) \in E(X), y_i \in V(Y)\},$$

$$B = \{((x, y_1), (x, y_2), \dots, (x, y_s)) : (y_1, y_2, \dots, y_s) \in E(Y), x \in V(X)\}.$$

Define the *wreath (or lexicographic) product* of X and Y to be the hypergraph $X \wr Y$ such that $V(X \wr Y) = V(X) \times V(Y)$ and $E(X \wr Y) = A \cup B$. Define the *wreath product* of two transitive groups G_1 and G_2 acting faithfully on Ω_1 and Ω_2 to be the group of all $\gamma \in S_{\Omega_1 \times \Omega_2}$ such that

$$\gamma(i, j) = (g(i), h_i(j)),$$

where $g \in G_1$, and $h_i \in G_2$, and denote it by $G_1 \wr G_2$. Sabidussi [55] showed that, given a vertex-transitive graph Γ with $H \leq \text{Aut}(\Gamma)$, H transitive and $n =$

$|\text{Stab}_H(x)|$, for $x \in V(\Gamma)$, then $\Gamma \wr E_n$ is a Cayley graph of H , and if $\Gamma \neq E_k$ then $\text{Aut}(\Gamma \wr E_n) = \text{Aut}(\Gamma) \wr S_n$. In Chapter 4, we generalize Sabidussi's result to vertex-transitive hypergraphs.

Let X be a vertex-transitive hypergraph and $G \leq \text{Aut}(X)$ a minimal transitive subgroup. Let X' be another vertex-transitive hypergraph with $G \leq \text{Aut}(X')$ also a minimal transitive subgroup. Let $n = |\text{Stab}_G(x)|$, $x \in V(X)$. We say that X is an *n-Cayley hypergraph* of G . We will show that X is isomorphic to X' if and only if $X \wr E_n$ and $X' \wr E_n$ are isomorphic. Now, assume $X \wr E_n$ and $X' \wr E_n$ are isomorphic by $\alpha \in \text{Aut}(G)$. Then α permutes the left cosets of $\text{Stab}_G(x)$. Denote this permutation by α_* . Note that we can and do consider $V(X)$ and $V(X')$ as left cosets of $\text{Stab}_G(x)$, and denote the left cosets of $\text{Stab}_G(x)$ by G_* . We say that X is an *n-CI-hypergraph* of G if and only if whenever X' is a hypergraph as above, then X and X' are isomorphic if and only if they are isomorphic by some α_* , $\alpha \in \text{Aut}(G)$. Similarly, G is an *n-CI-group with respect to hypergraphs* if and only if every hypergraph X with $G \leq \text{Aut}(X)$, G a minimal transitive subgroup, is an *n-CI-hypergraph* and a *weak n-CI-group via G'* if and only if G is not an *n-CI-group* with respect to hypergraphs but every *n-Cayley hypergraph* of G that is not an *n-CI-hypergraph* of G is a Cayley hypergraph of G' and G' is an *n-CI-group* with respect to hypergraphs. In Chapter 4 we generalize Lemma 2.4 to the following result.

Lemma 4.16. The following are equivalent:

- (i) X is a *n-CI-hypergraph* of G ,

(ii) given a permutation $\phi \in S_{G_\bullet}$, whenever $\phi^{-1}G\phi \leq \text{Aut}(X)$, then $\phi^{-1}G\phi$ and G are conjugate in $\text{Aut}(X)$.

Also in Chapter 4 we prove using this result that if $\langle \rho, \tau \rangle$ is a nonabelian group of degree pq and order pq^n as above, $n \geq 1$, then $\langle \rho, \tau \rangle$ is a weak q^{n-1} -CI-group via \mathbb{Z}_{pq} with respect to graphs. We will then determine necessary and sufficient conditions for two vertex-transitive graphs of order pq with the same minimal transitive subgroup to be isomorphic. Combined with the characterization of vertex-transitive graphs mentioned above, this yields a classification of vertex-transitive graphs of order pq .

In [3], Alspach and Parsons observed that if Γ is a (q, p) -metacirculant graph, $q < p$ prime, and $p^2 \mid |\text{Aut}(\Gamma)|$, then Γ is isomorphic to a Cayley graph. If p^2 does not divide $|\text{Aut}(\Gamma)|$ and $q^2 \mid |\alpha|$, then Alspach and Parsons have characterized those (q, p) -metacirculant graphs that are Cayley graphs in the following fashion.

Theorem 2.6. Let $\Gamma = \Gamma(q, p, \alpha, S_0, \dots, S_\mu)$ be a metacirculant graph such that $q^2 \mid |\alpha|$ and $p^2 \nmid |\text{Aut}(\Gamma)|$. Then Γ is a Cayley graph if and only if $\alpha(S_0) = S_0$ and there exists a cyclic permutation $\sigma \in S_q$ and a sequence $a_0, a_1, \dots, a_q \in \mathbb{Z}_p$ having the properties,

(i) If $0 < r \leq \mu$, $0 \leq k \leq q-1$, and $\sigma(k+r) \equiv \sigma(k) + t \pmod{q}$ for some t such that $0 < t \leq \mu$, then

$$S_r = \alpha^{\sigma(k)-k} S_t + \alpha^{-k}(a_k - a_{k+1}) \pmod{p}.$$

(ii) If $0 < r \leq \mu$, $0 \leq k \leq q-1$ and $\sigma(k+r) \equiv \sigma(k+r) + t \pmod{q}$ for some t such that $0 < t \leq \mu$, then

$$S_r = -\alpha^{\sigma(k+r)-k} S_t + \alpha^{-k} (a_k - a_{k+r}) \pmod{p}.$$

(iii) For some $\gamma \in \mathbb{Z}_p$, where either $\gamma = 1$ or $\gamma = \alpha^{a/q}$, we have

$$\gamma^q a_{i0} + \gamma^{q-1} a_{i1} + \dots + \gamma a_{i(q-1)} \equiv 0 \pmod{p}.$$

Using techniques similar to those used in solving the above problems, we improve this characterization to the following result.

Corollary 4.2. Let $\Gamma = \Gamma(q, p, \alpha, S_0, \dots, S_\mu)$ where $p > q$. Then Γ is a Cayley graph if and only if $\Gamma = \Gamma(q, p, \alpha', S_0, \dots, S_\mu)$, $|\alpha'| = q$ or $|\alpha'| = 1$. Further, if $q^2 \mid |\alpha|$, then Γ is a Cayley graph if and only if Γ is circulant.

Lemma 2.4 and Lemma 4.16 point out the importance of being able to determine the automorphism groups of vertex-transitive graphs, or at least large subgroups of them. Determining automorphism groups of vertex-transitive graphs is an interesting problem in its own right, and is known as the König problem. In general, the problem is hopeless, as solving it presupposes knowing all finite groups. However, for specified families of vertex-transitive graphs, the problem is much more attemptable. In particular, for a circulant graph of order n , this problem has been solved using the method of Schur (see [58]) for $n = p^k$, p a prime by Klin and Pöschel [36], [37], and for square free n by Gol'fand [31]. In Chapter 3 we give an

original proof for $n = pq$, and in Chapter 5 for $n = p^k$, p and q distinct primes with p odd. For $n = p^k$, p an odd prime, the result follows easily from our earlier extension of Burnside's Theorem. We remark that Klin and Pöschel's proof of this result is essentially a special case of our extension of Burnside's Theorem. In Chapter 3 we also show that if Γ is a (q, p) -metacirculant graph that is not isomorphic to a circulant graph, then either $\text{Aut}(\Gamma) \leq \langle \rho, \tau \rangle$ for some choice of α , or $\text{Aut}(\Gamma)$ is primitive and one of a known list.

We would like to remark that the characterization of vertex-transitive graphs of order pq along with calculating the automorphism groups of metacirculant graphs of order pq , solves a long standing problem in permutation group theory. We have already seen that if G is a primitive group, then there exists a graph Γ such that $G \leq \text{Aut}(\Gamma)$. Let H be a *regular group*, that is, a group whose order and degree are equal. We say H is a *Burnside group* if whenever G is a group that contains H as a regular subgroup, then G is either doubly transitive or imprimitive. For abelian groups, we have the following useful result [58].

Theorem 2.7. Let H be an abelian group with cyclic Sylow p -subgroup. Then H is a Burnside group.

For nonabelian groups of order pq and special values of p and q , some results on which groups are Burnside groups are known (see [58]). However, in general, this problem was open. By the comments above, to solve this problem, we need

only check which (q, p) -metacirculant graphs which are also Cayley graphs have a primitive automorphism group. This was done by Praeger and Xu [52].

Note that we may generalize this question to ask the following. Let H be a transitive group of degree m acting on Ω , that contains no nontrivial transitive subgroups, and let $n = |\text{Stab}_H(x)|$, $x \in \Omega$. We say that H is an n -Burnside group if whenever G is a group that contains H as a minimal transitive subgroup then G is doubly transitive or imprimitive. For groups H of degree pq that contain no nontrivial transitive subgroups and are not simple, we need only determine which (q, p) -metacirculant graphs which are not Cayley graphs have primitive automorphism groups. This is done in Chapter 3.

We would like to point out that the above work on automorphism groups of vertex-transitive graphs together with its applications to permutation group theory raises some interesting questions in permutation group theory. As stated above, in Chapter 3 we determine which transitive graphs of degree pq are n -Burnside groups. An obvious question that arises from this result is to determine those groups G of degree n that can occur as the minimal transitive subgroup of some permutation group H of degree n .

Let ρ, τ be as above with $m = q$, $n = p$, and $|\alpha| = q^i$, $i \geq 1$. We have already seen that if G is a group of degree pq with $\langle \rho, \tau \rangle \leq G$, then there exists a (q, p) -metacirculant graph Γ with $G \leq \text{Aut}(\Gamma)$. Further, $\text{Aut}(\Gamma)$ is known, and if Γ is not a circulant graph, then one can easily show that $\text{Aut}(\Gamma)$ is primitive or $\langle \rho, \tau \rangle \triangleleft \text{Aut}(\Gamma)$. Note that in the latter case $\langle \rho, \tau \rangle$ is generated by the union of a

Sylow p -subgroup of $\text{Aut}(\Gamma)$ and a Sylow q -subgroup of $\text{Aut}(\Gamma)$. This shows that Burnside's Theorem can be extended to permutation groups of non-prime power degree in a natural fashion. Another natural question would thus be determine the weakest necessary conditions possible for a group H of degree pq , generated by the union of a q -subgroup and a p -subgroup to have the property that if $H \leq G$ is minimally transitive, then G is primitive or $H \triangleleft G$.

We now define some terms and give some notation that will be used throughout the work on algebraic graph theory in this dissertation. Let G be a transitive permutation group of degree n that admits a complete block system \mathcal{B} of m blocks of size k , $mk = n$. For each $g \in G$, g induces a permutation contained in S_m of the blocks of \mathcal{B} . We denote this induced permutation by g/\mathcal{B} . If \mathcal{B} is the unique complete block system of G of m blocks of size k , we sometimes denote the induced permutation g/\mathcal{B} by g/k .

Let Γ be a vertex-transitive graph of order n such that there exists $G \leq \text{Aut}(\Gamma)$ such that G admits a complete block system \mathcal{B} of m blocks of size k , $mk = n$. Define a graph Γ/\mathcal{B} by $V(\Gamma/\mathcal{B}) = \mathcal{B}$ and $E(\Gamma/\mathcal{B}) = \{B_i B_j : B_i, B_j \in \mathcal{B} \text{ and some vertex of } B_i \text{ is adjacent to some vertex of } B_j\}$. As above, if \mathcal{B} is the unique block system of G consisting of m blocks of size k , then we sometimes denote Γ/\mathcal{B} by Γ/k .

Let G be a transitive permutation group acting faithfully on a set Ω . We define the *2-closure* of G , denoted $\text{cl}(G)$, to be the largest subgroup of S_Ω such that the orbits of $\text{cl}(G)$ acting on $\Omega \times \Omega$ are the same as the orbits of G acting on $\Omega \times \Omega$.

Hence if $\Gamma_1, \dots, \Gamma_s$ are the orbital digraphs of G , then $\text{cl}(G) = \cap_{i=1}^s \text{Aut}(\Gamma_i)$. Note that if G is doubly transitive, then $\text{cl}(G) = S_\Omega$.

CHAPTER 3

AUTOMORPHISM GROUPS OF VERTEX-TRANSITIVE GRAPHS OF ORDER A PRODUCT OF PRIMES

In this chapter, we will determine most of the automorphism groups of vertex-transitive graphs of order pq . We will determine the automorphism groups of circulant graphs of order pq and (q, p) -metacirculant graphs that are not circulant graphs. Note that the automorphism groups of circulant graphs of order pq have already been determined [31], [36]. As a corollary, we will then have the automorphism groups of all vertex-transitive graphs of order pq , with the exception of a class discovered by Marušič and Scapellato [47] where $p = 2^{2^s} + 1$ and $q | 2^{2^s} - 1$. Hence in that case, p is a Fermat prime. As a corollary of that, we will determine the groups of order pq that are not Burnside groups, giving an explicit list of those groups that are not along with their maximal non-doubly transitive over groups. We also generalize the notion of a Burnside group and determine which metacirculant groups G of order pq have the property that if G is the minimal transitive subgroup of H , H a transitive group of degree pq , then H is imprimitive.

The first lemma that we will prove is fundamental to most of the work in algebraic graph theory contained in this dissertation. Clearly if Γ is a vertex-transitive graph, then the structure of $\text{Aut}(\Gamma)$ will impose some structure on Γ and vice versa. The first lemma we will prove essentially determines some of the structure imposed on $\text{Aut}(\Gamma)$ under certain assumptions. Before proceeding to the lemma, we need some additional notation.

A graph Γ is said to be an (m, p) -galactic graph if there exists $\alpha \in \text{Aut}(\Gamma)$ such that all of the orbits of α have order p , and $|V(\Gamma)| = mp$. Let $[\alpha]$ be the subgroup of $\text{Aut}(\Gamma)$ such that if $\delta \in [\alpha]$, then the orbits of $\delta^{-1}\alpha\delta$ are the same as the orbits of α . A graph Γ will be called an (m, p) -uniformly galactic graph if Γ is an (m, p) -galactic graph and $[\alpha]$ is transitive.

Let G be a transitive permutation group that admits a complete block system $\mathcal{B} = \{B_i : i \in \mathbb{Z}_m\}$ of m blocks of size p , p a prime, and \mathcal{B} is formed by the orbits of some normal subgroup $N \triangleleft G$. Then for each B_i there exists $\alpha_i \in N$ such that $\alpha_i|_{B_i}$ is a p -cycle. Define an equivalence relation \equiv on the blocks B_0, \dots, B_{m-1} by $B_i \equiv B_j$ if and only if whenever $\alpha \in N$ then $\alpha|_{B_i}$ is a p -cycle if and only if $\alpha|_{B_j}$ is a p -cycle. Denote the equivalence classes of \equiv by C_0, \dots, C_a and let $E_i = \cup_{j \in C_i} B_j$. Then

Lemma 3.1. Let Γ be a vertex-transitive graph with $G \leq \text{Aut}(\Gamma)$ as above. Then there exists $H \leq \text{Aut}(\Gamma)$ such that $G \leq H$ and each E_i is a block of H . Further, Γ is a (m, p) -uniformly galactic graph and for each $0 \leq i \leq a$ there exists $\alpha_i \in H$ such that $\alpha_i|_{E_i}$ is semiregular of order p and $\alpha_i|_{E_j} = 1$ for every $i \neq j$.

Proof. We first show that if $B_j \in E_i$, $B_k \notin E_i$ and some vertex of B_j is adjacent to some vertex of B_k , then every vertex of B_j is adjacent to every vertex of B_k . This will imply that for each equivalence class E_i that there exists $\alpha_i \in \text{Aut}(\Gamma)$ such that $\alpha_i|_{B_s}$ is a p -cycle for every $B_s \in E_i$ and $\alpha_i|_{B_t} = 1$ for every $B_t \notin E_i$, and

so that Γ is an (m, p) -uniformly galactic graph. We then show that each E_i is a block of $H = \langle G, \alpha_i : 0 \leq i \leq a \rangle$.

As $B_j \in E_i$ and $B_k \notin E_i$, there exists $\gamma_j \in G$ such that either $\gamma_j|_{B_j}$ is a p -cycle and $\gamma_j|_{B_k}$ is not, or $\gamma_j|_{B_k}$ is a p -cycle and $\gamma_j|_{B_j}$ is not. Without loss of generality assume that $\gamma_j|_{B_j}$ is a p -cycle and $\gamma_j|_{B_k} = 1$. Let $\delta_k \in G$ such that $\delta_k|_{B_k}$ is a p -cycle. If $\delta_k|_{B_j}$ is not a p -cycle, then we assume without loss of generality that $\delta_k|_{B_j} = 1$. Then $\gamma_j\delta_k|_{B_j}$ is a p -cycle and $\gamma_j\delta_k|_{B_k}$ is a p -cycle. We conclude that each vertex of B_j is adjacent to some vertex of B_k . Further, as $\gamma_j \in G$, each vertex of B_k is adjacent to every vertex of B_j , and similarly, as $\delta_k \in G$ every vertex of B_k is adjacent to every vertex of B_j . If $\delta_k|_{B_j}$ is a p -cycle, then each vertex of B_k is adjacent to some vertex of B_j . As $\gamma_j \in G$, each vertex of B_k is adjacent to every vertex of B_j . Hence every vertex of B_j is adjacent to every vertex of B_k .

Thus for each equivalence class E_i there exist $\alpha_i \in \text{Aut}(\Gamma)$ such that $\alpha_i|_{B_s}$ is a p -cycle for every $B_s \in E_i$ and $\alpha_i|_{B_t} = 1$ for every $B_t \notin E_i$. Suppose $\beta \in H$ such that $\beta(E_i) \cap E_i \neq \emptyset$ and $\beta(E_i) \neq E_i$. Then there exists $B_s \in E_i$ such that $\beta(B_s) \notin E_i$ and $B_t \in E_i$ such that $\beta(B_t) \in E_i$. Then $\beta\alpha_i\beta^{-1}|_{\beta(B_t)}$ is a p -cycle and there exists $B_u \in E_i$ such that $\beta\alpha_i\beta^{-1}|_{B_u} = 1$, a contradiction. Hence each E_i is a block of H . \square

Before proceeding to the main problems of this chapter, we first give an independent argument for a result of Marušič [45] giving a necessary and sufficient

condition for a vertex-transitive graph of order pq to be isomorphic to a (q, p) -metacirculant graph.

Theorem 3.2. Let X be a vertex-transitive combinatorial object of order pq and $G \leq \text{Aut}(\Gamma)$ a transitive subgroup that admits a complete block system \mathcal{B} of q blocks of size p . Define $\pi : G \rightarrow S_q$ by $\pi(\gamma) = \gamma/\mathcal{B}$. If $\text{Ker}(\pi) \neq 1$ and $p^2 \nmid |\text{Ker}(\pi)|$, then X is isomorphic to a metacirculant combinatorial object.

Proof. As $\text{Ker}(\pi) \neq 1$ and $\text{Ker}(\pi) \triangleleft G$, \mathcal{B} is formed by the orbits of $\text{Ker}(\pi)$. Hence $\text{Ker}(\pi)|_B$ is a transitive group for every $B \in \mathcal{B}$. As p is prime, there exists $\rho \in \text{Ker}(\pi)$ such that $|\rho|_B| = p$, for some $B \in \mathcal{B}$, and we may assume without loss of generality that $|\rho| = p$. Let $\tau_1 \in G$ such that $|\pi(\tau_1)| = q$. Then $\langle \rho \rangle$ and $\langle \tau_1^{-1} \rho \tau_1 \rangle$ are Sylow p -subgroups of $\text{Ker}(\pi)$, and hence there exists $\delta \in \text{Ker}(\pi)$ such that $\delta^{-1} \langle \tau_1^{-1} \rho \tau_1 \rangle \delta = \langle \rho \rangle$. Let $\tau = \tau_1 \delta$. Then $\tau^{-1} \rho \tau = \rho^\alpha$ for some α , and by [19] $\langle \rho, \tau \rangle$ is isomorphic to a metacirculant group. \square

Corollary 3.3. Let Γ be a vertex-transitive graph of order pq . Then Γ is isomorphic to a metacirculant graph if and only if there exist $G \leq \text{Aut}(\Gamma)$ such that G admits a complete block system \mathcal{B} of q blocks of size p , and $\text{Ker}(\pi) \neq 1$, where π is defined as above.

Proof. As $\langle \rho, \tau \rangle \leq \text{Aut}(\Gamma)$ for every metacirculant graph Γ , every metacirculant graph contains such a subgroup G . Conversely, if Γ is a vertex-transitive graph with G as above, then if $p^2 \nmid |\text{Ker}(\pi)|$, then the result follows from Theorem 3.2.

If $p^2 \mid |\text{Ker}(\pi)|$, then by Lemma 3.1, Γ is the wreath product of a circulant graph of order q over a circulant graph of order p , and is thus isomorphic to a circulant graph. \square

We now prove some group theoretic results which will be needed in the main results of this chapter.

Let $n \in \mathbb{Z}$ and $\alpha \in S_n$ such that there exists $f \in \mathbb{Z}_n[x]$ and $\alpha(x) = f(x)$ for all $x \in \mathbb{Z}_n$. Such a permutation will be called a *permutation polynomial* on \mathbb{Z}_n , and P_n will denote the group of all permutation polynomials on \mathbb{Z}_n . The following was first proven in [20].

Lemma 3.4. Let $n = p_1 p_2 \dots p_r$, where $p_1 < p_2 < \dots < p_r$ are prime. Then $P_n \cong S_{p_1} \times S_{p_2} \times \dots \times S_{p_r}$.

Proof. Define a function $f : P_n \rightarrow S_{p_1} \times \dots \times S_{p_r}$ by

$$f(g) = (g \bmod p_1, g \bmod p_2, \dots, g \bmod p_r).$$

By [17], this function is well defined and $f(gh) = f(g)f(h)$ for all $g, h \in P_n$. Thus f is a homomorphism. We first show that f is surjective.

Let $h \in S_{p_1} \times \dots \times S_{p_r}$. Then for each $1 \leq i \leq r$, there exists $h_i \in S_{p_i}$ ([50], pg 53) such that $h = h_1 \times h_2 \times \dots \times h_r$ and $h_i(x) = \sum_{j=0}^{p_i-1} a_{i,j} x^j$. For each i, j , let $b_{i,j} \in \mathbb{Z}_n$ such that $b_{i,j} \equiv a_{i,j} \bmod p_i$ and $b_{i,j} \equiv 0 \bmod n/p_i$, and

let $h'_i = \sum_{j=0}^{p_i-1} b_{i,j} x^j$. Let $h' : \mathbb{Z}_n \rightarrow \mathbb{Z}_n$ by $h' = \sum_{i=1}^r h'_i$. Clearly $h' \in \mathbb{Z}_n[x]$. Further, let $a, b \in \mathbb{Z}_n$, $a \neq b$. Then there exists a prime p_i such that $a \not\equiv b \pmod{p_i}$ and as $h'(a) \equiv h_i(a) \pmod{p_i}$, $h(a) \equiv h_i(a) \pmod{p_i}$, $h'(b) \equiv h(b) \pmod{p_i}$, we have $h'(a) \not\equiv h'(b) \pmod{p_i}$. Thus $h'(a) \neq h'(b)$ and so h' is a bijection, $h' \in P_n$. Clearly $f(h') = h$ and so f is surjective.

Let $h \in \text{Ker}(f)$. Then $h(x) \equiv x \pmod{p_i}$ for all $x \in \mathbb{Z}_n$, $1 \leq i \leq r$. By the Chinese Remainder Theorem, h is the identity permutation, $\text{Ker}(f) = 1$, f is injective and so f is an isomorphism. \square

Lemma 3.5. Let $n = p_1 p_2 \dots p_r$ be square free. Then a permutation group G of order n admits a complete block system of m blocks each of size n/m for every nontrivial divisor m of n if and only if $G \cong S$, where $S \leq S_{p_1} \times S_{p_2} \times \dots \times S_{p_r}$.

Proof. By Lemma 3.4, $S \leq S_{p_1} \times S_{p_2} \times \dots \times S_{p_r}$ if and only if $S \cong S' \leq P_n$. Let $B_j = \{i \in \mathbb{Z}_n : i \equiv j \pmod{n/m}\}$. Let $\beta \in S'$. Then by [17] pg. 72, $\beta(B_j) = B_k$ for some k . Hence S' admits a complete block system of m blocks of size n/m .

Conversely, assume S' admits a complete block system of m blocks of size n/m for every nontrivial divisor m of n . We proceed by induction on the number of divisors of n . If n has two prime divisors, then $S'/q \leq S_p$ and $S'/p \leq S_q$ are both defined. It follows by the Chinese Remainder Theorem that $S' \leq S'/p \times S'/q \leq S_q \times S_p$.

Let $s \geq 2$ and assume the result is true for all square free n with s prime factors. Let n be square free with $s + 1$ prime factors, and assume that S' is a transitive group of degree n that admits a complete block system of m blocks of size n/m for every nontrivial divisor m of n . Let $p_1 | n$. Then S' admits a complete block system of $p_2 \dots p_{s+1} = n/p_1$ blocks of size p_1 and p_1 blocks of size $p_2 \dots p_{s+1}$. By the induction hypothesis and the Chinese Remainder Theorem, $S' \leq S'/p_1 \times S'/(n/p_1) \leq S_{p_1} \times S_{p_2} \times \dots \times S_{p_{s+1}}$. Hence the result follows by induction. \square

3.1 CIRCULANT GRAPHS

The preceding lemma determines the structure of many automorphism groups of circulant graphs of order pq . We will show that if Γ is circulant of order pq and $\Gamma \neq K^{pq}, E^{pq}$, or a wreath product, that, in fact, $\text{Aut}(\Gamma) \leq S_q \times S_p$. This will allow us to determine strong constraints on $\text{Aut}(\Gamma)$. We remark that automorphism groups of symmetric graph Γ of order pq (Γ is *symmetric* if $\text{Aut}(\Gamma)$ is transitive and also transitive on the set of adjacent vertices of Γ) have been investigated (see [51]).

Let $V^i = \{v_j^i : j \in \mathbb{Z}_p\}$, and $V_j = \{v_j^i : i \in \mathbb{Z}_q\}$. If Γ is circulant and $\text{Aut}(\Gamma)$ admits a complete block system of q blocks of size p , then the blocks are the sets V^i , $i \in \mathbb{Z}_q$, and if $\text{Aut}(\Gamma)$ admits a complete block system of p blocks of size q , the blocks are the sets V_j , $j \in \mathbb{Z}_p$. For an integer m , let $N(m) = \{f : \mathbb{Z}_m \rightarrow \mathbb{Z}_m \text{ where } f(x) = ax + b, a \in \mathbb{Z}_m^*, b \in \mathbb{Z}_m\}$.

Theorem 3.6. Let Γ be a circulant graph of order pq . Then one of the following assertions is true:

- (i) if $\text{Aut}(\Gamma)$ is doubly transitive, then $\text{Aut}(\Gamma) = S_{pq}$,
- (ii) if $\text{Ker}(\pi_2) \neq 1$, then $\text{Aut}(\Gamma) = \text{Aut}(\Gamma/p) \wr \text{Aut}(\Gamma[V^0])$,
- (iii) if $\text{Ker}(\pi_2) = 1$ and $\text{Ker}(\pi_1) \not\cong \mathbb{Z}_p$, then $\text{Aut}(\Gamma) \leq S_p \times S_q$, and there exists $K \triangleleft \text{Aut}(\Gamma)$ and $K' \triangleleft \text{Aut}(\Gamma)$ such that the orbits of K and K' have order p and q respectively. Then if $K|_{V^0}$ and $K'|_{V_0}$ are doubly transitive, then $\text{Aut}(\Gamma) = S_p \times S_q$. If $K|_{V^0}$ is doubly transitive and $K'|_{V_0}$ is not, then $\text{Aut}(\Gamma) = S_p \times A$, $A < N(q)$. If $K'|_{V_0}$ is doubly transitive and $K|_{V^0}$ is not, then $\text{Aut}(\Gamma) = S_q \times B$, $B < N(p)$. If neither $K|_{V^0}$ nor $K'|_{V_0}$ are doubly transitive, then $\text{Aut}(\Gamma) \leq A \times B$, where A and B are as above.

Proof. Let ρ and τ be defined as in Chapter 2, with $\alpha = 1$. If Γ is circulant, then by Theorem 2.6 $\text{Aut}(\Gamma)$ is either doubly transitive or imprimitive. If $\text{Aut}(\Gamma)$ is doubly transitive, then clearly $\Gamma = K^{pq}$ or E^{pq} and $\text{Aut}(\Gamma) = S_{pq}$. If $\text{Aut}(\Gamma)$ is imprimitive, $\text{Aut}(\Gamma)$ admits a complete block system \mathcal{B} of, say, q blocks each of size p , where the blocks are formed by the orbits of ρ . By comments above, the blocks of $\text{Aut}(\Gamma)$ of size p are the sets V^i , $i \in \mathbb{Z}_q$. As Γ is circulant, we may thus take H of Corollary 3.3 to be $\text{Aut}(\Gamma)$. Let K be a maximal normal subgroup of $\text{Aut}(\Gamma)$ such that K is not transitive and $\rho \in K$. Define $\pi_1 : \text{Aut}(\Gamma) \rightarrow S_q$ by $\pi_1(\gamma) = \gamma/\mathcal{B}$. Then $\text{Ker}(\pi_1) = K$. Define $\pi_2 : K \rightarrow S_p$ by $\pi_2(\gamma) = \gamma|_{B_0}$, $B_0 \in \mathcal{B}$

such that $v_0^0 \in B_0$. We first show that if $\text{Ker}(\pi_2) \neq 1$, then $\Gamma = (\Gamma/p) \wr (\Gamma[V^0])$ and $\text{Aut}(\Gamma) = \text{Aut}(\Gamma/p) \wr \text{Aut}(\Gamma[V^0])$.

Let $\beta \in K$ such that $\beta \in \text{Ker}(\pi_2)$. Hence for some i , $\beta|_{V^i} \neq 1$. Note that for every $\delta \in K$, $\delta^{-1}\beta\delta \in \text{Ker}(\pi_2)$. If $K|_{V^0}$ is doubly transitive, then there exists $a, b \in V^i$ such that $\beta(a) = b$. As $K|_{V^0}$ is doubly transitive and $K|_{V^j} = K|_{V^0}$ for every $j \in \mathbb{Z}_q$, for each $c \in V^i$ such that $c \neq a$, there exists $\delta_c \in K|_{V^i}$ such that $\delta_c(a) = a$ and $\delta_c(b) = c$. Then $\delta_c\beta\delta_c^{-1}(a) = c$. As for each $d \in V^i$ there exists $\gamma_d \in \text{Ker}(\pi_2)$ such that $\gamma_d(d) \neq d$, we conclude that $\text{Ker}(\pi_2)|_{V^i}$ is transitive and thus contains a p -cycle. It follows Lemma 3.1 that the Sylow p -subgroups of K have order p^{q+1} and that Γ is the wreath product of an order q circulant over an order p -circulant, $\Gamma = (\Gamma/p) \wr (\Gamma[V^0])$. By [54], $\text{Aut}(\Gamma) = \text{Aut}(\Gamma/p) \wr \text{Aut}(\Gamma[V^0])$.

If $K|_{V^i}$ is not doubly transitive, then if $\beta|_{V^i}$ is a p -cycle, then by the above argument Γ is isomorphic to a circulant graph. If $\beta|_{V^i}$ is not a p -cycle then, by Theorem 2.6, $K|_{V^i} \leq N(p)$, and so

$$\beta^{-1}z_0z_1 \dots z_{q-1}\beta = z_0z_1^{a_1}z_2^{a_2} \dots z_{q-1}^{a_{q-1}} \in \text{Ker}(\pi_2),$$

where $z_k(v_j^i) = v_j^i$, $i \neq k$, $z_k(v_j^i) = v_{j+1}^i$ if $i = k$, and each $a_k \neq 0$, $a_t \neq 1$. Hence

$$(z_0z_1 \dots z_{q-1})^{p-1}z_0z_1^{a_1} \dots z_{q-1}^{a_{q-1}} = z_1^{a_1+p-1} \dots z_{q-1}^{a_{q-1}+p-1} \in \text{Ker}(\pi_2),$$

and as $a_i + p - 1 \not\equiv 0 \pmod{p}$, Γ is again the wreath product of an order q circulant over an order p circulant. Thus $\text{Aut}(\Gamma) = \text{Aut}(\Gamma/p) \wr \text{Aut}(\Gamma[V^0])$.

If $\text{Ker}(\pi_2) = 1$, then either $\text{Ker}(\pi_1) \cong \mathbb{Z}_p$ or $\text{Stab}_K(v_0^0) \neq 1$. If $\text{Stab}_K(v_0^0) \neq 1$, then define an equivalence relation \equiv on V by $v_j^i \equiv v_\ell^k$ if and only if $\text{Stab}_K(v_j^i) = \text{Stab}_K(v_\ell^k)$. Note $K \cong J \cong 1_{S_q} \times J$, where $J \leq S_p$. As $K \cong 1_{S_q} \times J$, if $\text{Stab}_K(v_0^0) \neq 1$, there are p equivalence classes of \equiv and each has cardinality q , i.e. if E is an equivalence class of \equiv then E contains exactly one element from each orbit of K . Denote the equivalence classes of \equiv by E_0, E_1, \dots, E_{q-1} . Note that the equivalence classes E_0, E_1, \dots, E_{p-1} are blocks of $\text{Aut}(\Gamma)$ of size q , so we may assume that $v_0^0 \in E_0$ and $\rho(E_i) = E_{i+1}$. Hence $E_j = V_j$ for all $j \in \mathbb{Z}_p$. Then $\tau \in \text{Aut}(\Gamma)$ and the blocks $V_j, j \in \mathbb{Z}_p$ are formed by the orbits of τ . As $\rho, \tau \in \text{Aut}(\Gamma)$, it follows by Lemma 3.3 that $\text{Aut}(\Gamma) \leq S_q \times S_p$.

If $K|_{V^0}$ is doubly transitive then $\text{Stab}_K(v_j^i)$ is transitive on $V^i - \{v_j^i\}$, and if $v_j^i \in V^i$ and V^j is any other block of size p , then v_j^i is adjacent to $0, 1, p-1$, or p vertices of V^j , where v_j^i is adjacent to one vertex of V^k if and only if v_j^i is adjacent to only v_j^k , and v_j^i is adjacent to $p-1$ vertices of V^k if and only if v_j^i is adjacent to every vertex of V^k except v_j^k . Consider the function $\gamma : V \rightarrow V$ by $\gamma(v_j^i) = v_{\delta(j)}^{\sigma(i)}$, where $\sigma \in \text{Aut}(\Gamma)/p$ and $\delta \in S_p$. Note that, as $\sigma \in \text{Aut}(\Gamma)/p$, if V^r and V^s are blocks of size p and each vertex of V^r is adjacent to t vertices of V^s , then each vertex of $V^{\sigma(r)}$ is adjacent to t vertices of $V^{\sigma(s)}$. It follows from the comments above that $\gamma \in \text{Aut}(\Gamma)$. Let K' be a maximal normal subgroup of $\text{Aut}(\Gamma)$ that contains τ and is not transitive. Clearly $K' \neq 1$ as $\tau \in K'$. Thus if $K'|_{V_0}$ is not doubly transitive and $K|_{V^0}$ is, then $\text{Aut}(\Gamma) = A \times S_p, A \leq N(q)$. By analogous arguments, if $K'|_{V_0}$ is doubly transitive but $K|_{V^0}$ is not, then $\text{Aut}(\Gamma) = S_q \times B, B \leq N(p)$, and if both

$K|_{V^0}$ and $K'|_{V_0}$ are doubly transitive, $\text{Aut}(\Gamma) = S_q \times S_p$. Note that we must have $A < N(p)$, $B < N(p)$, as if r is a prime, $N(r)$ is itself doubly transitive. Otherwise, $\text{Aut}(\Gamma) < N(q) \times N(p)$.

If $\text{Ker}(\pi_1) = \mathbb{Z}_p$, then if $\text{Aut}(\Gamma)/p \leq N(q)$, then $\text{Aut}(\Gamma)/p$ contains a normal subgroup of order q , and hence $\text{Aut}(\Gamma)$ contains a normal subgroup of order pq [34], pg 45. Hence $\langle \rho, \tau \rangle \triangleleft \text{Aut}(\Gamma)$ so that $\text{Aut}(\Gamma) \leq N(pq) = N(p) \times N(q)$.

If $\text{Aut}(\Gamma)/p \not\leq N(q)$, then $\text{Aut}(\Gamma)/p$ is doubly transitive and there are at least two Sylow q -subgroups of $\text{Aut}(\Gamma)/p$. Hence, there exists $\beta \in \text{Aut}(\Gamma)$ such that β/p is a q -cycle, but $\langle \beta/q \rangle \not\leq \langle \tau/q \rangle$. Without loss of generality, we assume that $|\beta| = q$. As $\text{Ker}(\pi) = \langle \rho \rangle$, $\beta(v_j^i) = v_{\alpha j + b_i}^{\sigma(i)}$, $\sigma \in S_q$, $\alpha \in \mathbb{Z}_p^*$, and $b_i \in \mathbb{Z}_p$. We first show that $\beta(v_j^i) = v_{j+b_i}^{\sigma(i)}$, $\sigma \in S_q$, $b_i \in \mathbb{Z}_p$.

Suppose $\beta(v_j^i) = v_{\alpha j + b_i}^{\sigma(i)}$, $\sigma \in S_q$, $\alpha \in \mathbb{Z}_p^*$, $c_i \in \mathbb{Z}_p$. As $|\beta| = q$, we must have $|\alpha| = q$, or $|\alpha| = 1$. If $q > p$, then $q \nmid |\mathbb{Z}_p^*| = p - 1$. Hence $\alpha = 1$. If $q < p$, suppose $|\alpha| = q$. Then for some $r \in \mathbb{Z}_q$, $\tau^r \beta/p$ has a fixed point and hence does not have order q . But $\tau^r \beta(v_j^i) = v_{\alpha j + c_i}^{\sigma(i)+r}$ so $q \mid |\tau^r \beta|$. Hence $q \mid |\text{Ker}(\pi_2)|$, a contradiction. Hence $|\alpha| = 1$. As $\rho \in \text{Aut}(\Gamma)$, we may assume $b_0 = 0$.

Suppose that $\beta(v_j^i) = v_{j+b_i}^{\sigma(i)}$, $\sigma \in S_q$, $b_i \in \mathbb{Z}_p$, and for some k , $b_k \neq 0$. Let $r = \sigma(k) - k$, and $\tau' = \tau^{-r} \beta$. Then $\tau'(v_j^i) = v_{j+b_i}^{\sigma(i)-r}$ and $\tau'(v_j^k) = v_{j+b_k}^k$. Denote the orbits of τ' by $\mathcal{O}_1, \dots, \mathcal{O}_s$. Note that if $t = |\tau'|/p$, then $\tau'^t = \rho^{tb_k}$ as $\text{Ker}(\pi_2) \cong \mathbb{Z}_p \cong \text{Ker}(\pi_1)$, and that as $\text{Im}(\pi_1)$ is doubly transitive, that if $\phi \in \text{Aut}(\Gamma)$ such that $\phi|_{V^i} = 1$ and $\phi|_{V^j}$ is a p -cycle, then Γ is the wreath product of an order q -circulant over an order p -circulant, and so $\text{Ker}(\pi_1) \not\cong \mathbb{Z}_p$. Thus whenever $\phi \in \text{Aut}(\Gamma)$ such

that $\phi(V^i) = V^i$ and $\phi(V^j) = V^j$, if $\phi|_{V^i}$ is a p -cycle, then $\phi|_{V^j}$ is also a p -cycle. Now, if $1 \leq j \leq s$, $|\mathcal{O}_j| = c_j p$ where c_j is the length of the orbit \mathcal{O}_j/p in τ'/p . For each orbit \mathcal{O}_i , let $T_i = \{j : V^j \subseteq \mathcal{O}_i\}$. If $q > p$ and $p \mid |\tau'/p|$, then for every orbit \mathcal{O}_i such that $p \mid |\mathcal{O}_i/p|$, $\sum_{j \in T_i} b_j \equiv 0 \pmod p$, as τ'^p is the identity on the block V^0 . If $p \nmid |\mathcal{O}_i|$, let $d_i = |\mathcal{O}_i/p|$. Then $\sum_{j \in T_i} b_j \equiv d_i b_k \pmod p$. Let $t \equiv \sum_{i=0, p \nmid d_i}^{q-1} d_i \pmod p$. Then

$$\begin{aligned} \sum_{i=0}^{q-1} b_i &= \sum_{j=1}^s (\sum_{i \in T_j} b_i) \\ &= \sum_{j=1, p \nmid d_j}^s (\sum_{i \in T_j} b_i) = b_k t \end{aligned}$$

Clearly $t, b_k \in \mathbb{Z}_p^*$, so $\sum_{i=0}^{q-1} b_i \not\equiv 0 \pmod p$. If $q < p$ or $p < q$ and $p \nmid |\tau'|$, then

$$\sum_{i=0}^{q-1} b_i = \sum_{j=1}^s (\sum_{i \in S_j} b_i) = b_k t \not\equiv 0 \pmod p.$$

as $t = q$. However, as $|\beta| = q$, $\sum_{i=1}^q b_i \equiv 0 \pmod p$, a contradiction. Hence if $\beta \in \text{Aut}(\Gamma)$, $|\beta/p| = q$ and $|\beta| = q$, then $\beta(v_j^i) = v_j^{\sigma(i)}$.

By the above comments, there exists $\beta \in \text{Aut}(\Gamma)$ that satisfies the immediately preceding conditions. By Lemma 3.3, $\langle \rho, \tau, \beta \rangle$ admits a complete block system of p blocks of size q where the blocks are formed by the orbits of β and τ . Define $\pi'_1 : \langle \rho, \tau, \beta \rangle \rightarrow S_p$ by $\pi'_1(\gamma) = \gamma/q$. Then $\text{Ker}(\pi'_1)$ is doubly transitive and by arguments above we conclude that if $\gamma \in \{\xi : V \rightarrow V \text{ by } \xi(v_j^i) = v_j^{\sigma(i)}, \sigma \in S_q\}$, then $\gamma \in \text{Aut}(\Gamma)$. Note however that the function $\iota : V \rightarrow V$ by $\iota(v_j^i) = v_{-j}^{-i}$ is in $\text{Aut}(\Gamma)$ as Γ is circulant and the function $\gamma : V \rightarrow V$ by $\gamma(v_j^i) = v_j^{-i}$ is also in

$\text{Aut}(\Gamma)$. Hence $\iota\gamma \in \text{Aut}(\Gamma)$, $\iota\gamma \in \text{Ker}(\pi_1)$, and $\iota\gamma$ is not semiregular of order p , a contradiction. \square

We now give a procedure for calculating the automorphism group of a circulant graph of order pq , solving, for $n = pq$, a problem posed by Elspas and Turner [28].

First observe that if a circulant graph Γ is labeled with elements of \mathbb{Z}_{pq} so that the function $\gamma : \mathbb{Z}_{pq} \rightarrow \mathbb{Z}_{pq}$ by $\gamma(i) = i + 1$ is in $\text{Aut}(\Gamma)$, then there exists a set T such that there is an edge between i and j if and only if $i - j \in T$. We call T the *connection set* of T . Also, as \mathbb{Z}_{pq} is cyclic, there exists unique subgroups C_p and C_q of \mathbb{Z}_{pq} of order p and q , respectively.

Let Γ be a circulant graph of order pq with connection set T . If $\Gamma = K^{pq}$ or E^{pq} , then $\text{Aut}(\Gamma) = S_{pq}$. Otherwise, by [3], Γ is the wreath product of an order q circulant over an order p circulant if and only if whenever $a \notin C_p$, then $a + C_p \cap T = \emptyset$ or $a + C_p \subseteq T$ for every coset $a + C_p$, $a \notin C_p$. Hence to determine if Γ is the wreath product of two circulant graphs of prime order, one must check whether or not $a + C_p \cap T = \emptyset$ or $a + C_p \subseteq T$, $b + C_q \cap T = \emptyset$ or $b + C_q \subseteq T$ for every coset $a + C_p$ and $b + C_q$, $a \notin C_p$, $b \notin C_q$. If Γ is the wreath product of two circulants of prime order, then $\Gamma = \Gamma/p \wr \Gamma[C_q]$ or $\Gamma = \Gamma/q \wr \Gamma[C_p]$. These groups can easily be determined by Theorem 2.6.

If Γ is not the wreath product of two circulant graphs of prime order, then by Theorem 3.5, $\text{Aut}(\Gamma) = A \times S_p$, $B \times S_q$, $S_p \times S_q$, or $\text{Aut}(\Gamma) < N(q) \times N(p)$. Choose a primitive root r of p and let $a \in \mathbb{Z}_{pq}^*$ such that $a \equiv 1 \pmod{q}$, $a \equiv r \pmod{p}$.

Then $\text{Aut}(\Gamma)/q$ is doubly transitive if and only if $f : \mathbb{Z}_{pq} \rightarrow \mathbb{Z}_{pq}$ by $f(x) = ax$ is in $\text{Aut}(\Gamma)$. Let s be a primitive root of q , and let $b \in \mathbb{Z}_{pq}^*$ such that $b \equiv s \pmod{q}$, $b \equiv 1 \pmod{p}$. Let $g : \mathbb{Z}_{pq} \rightarrow \mathbb{Z}_{pq}$ by $g(x) = bx$. Then $\text{Aut}(\Gamma) = A \times S_p$ if and only if $f \in \text{Aut}(\Gamma)$, $g \notin \text{Aut}(\Gamma)$, $\text{Aut}(\Gamma) = S_q \times B$ is and only if $f \notin \text{Aut}(\Gamma)$, $g \in \text{Aut}(\Gamma)$, and $\text{Aut}(\Gamma) = S_q \times S_p$ if and only if $f, g \in \text{Aut}(\Gamma)$. Then one only needs determine what A and B are.

Finally, if $\text{Aut}(\Gamma) < N(p) \times N(q)$, then if $\beta \in N(q) \times N(p)$ and $\beta(0) = 0$, $\beta \in \text{Aut}(\Gamma)$ if and only if whenever \mathcal{O} is a non-singleton orbit of β , then $\mathcal{O} \cap T = \emptyset$ or $\mathcal{O} \subseteq T$. Let U be the set of all such $\beta \in N(p) \times N(q)$. Then $\text{Aut}(\Gamma) = \langle U, \gamma \rangle$.

3.2 GRAPHS WITH IMPRIMITIVE AUTOMORPHISM GROUPS

We now determine the automorphism groups of (q, p) -metacirculant graphs that are not isomorphic to a circulant graph. Note that if Γ is a vertex-transitive graph of order pq with an imprimitive automorphism group, then Γ is isomorphic to a metacirculant graph of order pq or is one of the class mentioned above discovered by Marušič and Scapellato. Again, let $V^i = \{v_j^i : j \in \mathbb{Z}_p\}$, and $V_j = \{v_j^i : i \in \mathbb{Z}_q\}$.

Theorem 3.7. Let Γ be a vertex-transitive graph of order pq , $p > q$, that is not isomorphic to a circulant graph such that $\text{Aut}(\Gamma)$ is imprimitive. Then

(i) if $\text{Aut}(\Gamma)$ admits a complete block system of q blocks each of size p , then $\text{Aut}(\Gamma) \cong \langle \rho, \tau \rangle$ for some choice of α , or

(ii) if $\text{Aut}(\Gamma)$ admits only a complete block system of p blocks of size q , then $p = 2^{2^s} + 1$ is a Fermat prime, and q divides $2^{2^s} - 1$. Further, the minimal transitive

subgroup of $\text{Aut}(\Gamma)$ that admits only a complete block system of p blocks of size q is isomorphic to $SL(2, 2^s)$ and $\text{Aut}(\Gamma)$ is isomorphic to a subgroup of $\text{Aut}(SL(2, 2^s))$.

Proof. (i) If $\text{Aut}(\Gamma)$ admits a complete block system of q blocks of size p , then by [46] Γ is isomorphic to a (q, p) -metacirculant graphs. If Γ is a metacirculant graph of order pq , $p > q$, that is not isomorphic to a circulant graph, then Γ is a (q, p) -metacirculant and $q \mid |\alpha|$. If $\text{Aut}(\Gamma)$ admits a complete block system \mathcal{B} of q blocks of size p , then the blocks are formed by the orbits of ρ . Define $\pi_1 : \text{Aut}(\Gamma) \rightarrow S_q$ by $\pi_1(\gamma) = \gamma/\mathcal{B}$ and $\pi_2 : K \rightarrow S_p$ by $\pi_2(\gamma) = \gamma|_B$, $B \in \mathcal{B}$ and $K = \text{Ker}(\pi_1)$. As $\rho \in \text{Ker}(\pi_1)$, $\text{Ker}(\pi_1) \neq 1$. As Γ is not isomorphic to a circulant graph, $\text{Ker}(\pi_2) = 1$, and the Sylow p -subgroups of $\text{Aut}(\Gamma)$ have order p . If $K|_{V^0}$ is doubly transitive, we claim that Γ is isomorphic to a circulant graph. Observe that $\text{Stab}_K(v_j^i)|_{V^i}$ is transitive on $V^i - V_j$. We conclude that if v_j^i is adjacent to some vertex of V^0 then v_j^i is either adjacent to only v_j^0 , or v_j^i is adjacent to v_t^0 where $t \neq j$, and v_j^i is not adjacent to v_j^0 , or v_j^i is adjacent to every vertex of V^0 .

Let $T_i = \{j : v_j^i \text{ is adjacent to } v_0^0\}$, $0 \leq i \leq p-1$. It suffices to show that if $j \in T_i$ then whenever $a - b \equiv i \pmod{q}$ and $r - s \equiv j \pmod{p}$ then v_r^a is adjacent to v_s^b . Let $j \in T_i$. Then v_0^0 is adjacent to v_j^i . If $i = 0$ then $\Gamma(V^0) = K^q$, the complete graph on q -vertices. Hence if $a - b \equiv 0 \pmod{q}$ and $r - s \equiv j \pmod{p}$ then v_r^a is adjacent to v_s^b . If $i \neq 0$, let $a, b \in \mathbb{Z}_p$ and $r, s \in \mathbb{Z}_q$ such that $a - b \equiv i \pmod{q}$ and $r - s \equiv j \pmod{p}$. As v_j^i is adjacent to v_0^0 and $\tau^a \in \text{Aut}(\Gamma)$, some vertex of V^a is adjacent to some vertex of V^b . Further, by comments above, v_r^a is adjacent to either one vertex of V^b , $p-1$ vertices of V^b , or to each vertex of V^b . If v_r^a is

adjacent to each vertex of V_b then clearly v_r^a is adjacent to v_s^b . If v_r^a is adjacent to $p-1$ vertices of V^b then v_r^a is not adjacent to v_r^b . Hence $j \neq 0$ and we then have that v_r^a is adjacent to v_s^b . If v_r^a is adjacent to one vertex of V^b , then $j = 0$ and so $r = s$. Hence v_r^a is adjacent to v_s^b . Therefore we now have that Γ is circulant. Thus $K|_{V^0}$ is not doubly transitive.

Suppose $\text{Aut}(\Gamma)/p$ is not contained in $N(q)$. Then $\text{Aut}(\Gamma)/p$ is doubly transitive and contains at least two Sylow q -subgroups. Let $\gamma \in \text{Aut}(\Gamma)$ such that $\langle \gamma/p \rangle$ is a Sylow q -subgroup of $\text{Aut}(\Gamma)/p$ but $\gamma/p \notin \langle \tau/p \rangle$. Assume without loss of generality that $|\alpha| = q^k$, $k \geq 1$, where $\tau : V \rightarrow V$ by $\tau(v_j^i) = v_{\alpha_j^i+1}^{i+1}$, and $|\gamma| = q^\ell$, $\ell \geq 1$. Note that as $K|_{V^0}$ is not doubly transitive, $\gamma(v_j^i) = v_{\beta_j+b_i}^{\sigma(i)}$, $\sigma \in S_q$, $\beta \in \mathbb{Z}_p^*$, $b_i \in \mathbb{Z}_p$. Then for some $r \in \mathbb{Z}_q^*$, $\tau^r \gamma/p$ has a fixed point and

$$\tau^r \gamma(v_j^i) = v_{\alpha^r \beta_j + \alpha^r b_i}^{\sigma(i)+r}.$$

Let $m = |\tau^r \gamma/p|$. As $\gamma/p \notin \langle \tau/p \rangle$, $m > 0$. Then $(\tau^r \gamma)^m(v_j^i) = v_{(\alpha^r \beta)^m j + c_i}^i$, $c_i \in \mathbb{Z}_p$. Now, $|\alpha| = q^k$, $k \geq 1$, and \mathbb{Z}_p^* is cyclic. Hence if $|(\alpha^r \beta)^m| = q^{k+c}$, $c \geq 0$, there exists $s \in \mathbb{Z}$ such that $(\alpha^r \beta)^{ks} = \alpha$, in which case $(\tau^r \alpha)^{ms} \tau(v_j^i) = v_{j+d_i}^{i+1}$, $d_i \in \mathbb{Z}_p$. Let $\gamma' = (\tau^r \gamma)^{ms} \tau$. Then $|\gamma'| = q$ or $|\gamma'| = pq$. If $|\gamma'| = pq$, then clearly Γ is isomorphic to a circulant graph. If $|\gamma'| = q$, then $\langle \gamma' \rangle \triangleleft \langle \rho, \gamma' \rangle$, $\langle \rho \rangle \triangleleft \langle \rho, \gamma' \rangle$, and $\langle \gamma' \rangle \cap \langle \rho \rangle = 1$. Thus $\langle \gamma', \rho \rangle \cong \mathbb{Z}_{pq}$, and so Γ is again isomorphic to a circulant graph. Thus $|(\alpha^r \beta)^m| = q^n$, $n < k$, $(\alpha^r \beta)^m = \alpha^t$, for some t , where $q|t$, and $\beta = \alpha^s$, for some s . Then

$$(\alpha^r \beta)^m = \alpha^{rm} \beta^m = \alpha^{rm} \alpha^{sm} = \alpha^t,$$

and so $rm + sm \equiv 0 \pmod{q}$. As $(m, q) = 1$, $r \equiv -s \pmod{q}$. Now, as $\gamma/p \notin \langle \tau/p \rangle$, there exists $r' \in \mathbb{Z}_q^*$, $r' \neq r$ such that $\tau^{r'} \gamma$ also has a fixed point and $0 \neq k' = |\tau^{r'} \gamma|$. By the argument above, $r' \equiv -s \pmod{q}$, and so $r \equiv r' \pmod{q}$, a contradiction. Thus $\text{Aut}(\Gamma)/p \leq N(q)$.

We now show that if $\text{Aut}(\Gamma)/p \leq N(q)$, then $\text{Aut}(\Gamma)/p = C_q$, where C_q is the unique Sylow q -subgroup of $N(q)$. Suppose there exists $\gamma \in \text{Aut}(\Gamma)$ such that $\gamma/p \notin C_q$. Then $\gamma(v_j^i) = v_{\beta j + b_i}^{\delta i + c}$, $\delta \in \mathbb{Z}_q^*$, $\delta \neq 1$, $c \in \mathbb{Z}_q$, $\beta \in \mathbb{Z}_p^*$, $b_i \in \mathbb{Z}_p$. Elementary calculations will show that if $\gamma' = \gamma^{-1} \tau^{-1} \gamma \tau$, then

$$\gamma'(v_j^i) = v_{j + \beta^{-1}(\alpha^{-1} b_{i+1} - b_{i+1 - \delta - 1})}^{i+1 - \delta^{-1}}.$$

As $\delta \neq 1$, $\delta^{-1} \neq 1$ and $|\gamma'| = q$ or $|\gamma'| = pq$, and by arguments above, Γ is isomorphic to a circulant graph.

Hence if $\gamma \in \text{Aut}(\Gamma)$, $\gamma(v_j^i) = v_{\beta j + b_i}^{i+r}$, $r \in \mathbb{Z}_q$, $\beta \in \mathbb{Z}_p^*$, $b_i \in \mathbb{Z}_p$. Then

$$\tau^{-r} \gamma(v_j^i) = v_{\alpha^{-r} \beta j + \alpha^{-r} b_i}^i.$$

If $\alpha^{-r} \beta = 1$, then if $b_i \neq b_j$ for some i, j , the Sylow p -subgroups of $\text{Aut}(\Gamma)$ have order at least p^2 , and so Γ is isomorphic to a circulant graph. If $\alpha^{-r} \beta \neq 1$, then $\text{Stab}_K(v_0^0) \neq 1$, and the equivalence relation \equiv is defined. Hence $\text{Aut}(\Gamma)$ admits a

complete block system of p blocks of size q , where the blocks are the equivalence classes of \equiv . As $\tau \in \text{Aut}(\Gamma)$, these equivalence classes must be the sets $V_j = \{v_j^i : i \in \mathbb{Z}_q\}$. Then $b_i = b_j$ for all i, j . We conclude that $\text{Aut}(\Gamma) \leq C_q \times A$, $A < N(p)$.

Let $|\alpha| = q^k$, $k \geq 1$. Let $\tau' \in \text{Aut}(\Gamma)$ such that $|\tau'| = q^\ell$, $\ell \geq 1$, and $\tau'(v_j^i) = v_{\beta_j}^{\sigma(i)}$, where $\sigma(i) = i + r$ or $\sigma(i) = i$, and $|\beta| = q^m$. We claim that $\tau' \in \langle \tau \rangle$. Note that this will imply the result, as if the claim is true, then as $\text{Aut}(\Gamma) \leq C_q \times A$, $A \leq N(p)$, then there exist $\omega \in A$ such that $C_q \times \omega = \tau_1$ and if $A' = \langle A - \langle \omega, C_p \rangle \rangle$, then A' is cyclic and is thus generated by some $\gamma \in A'$. Then $\tau(v_j^i) = v_{\omega\gamma_j}^{i+1}$ is in $\text{Aut}(\Gamma)$, $\text{Aut}(\Gamma) \leq \langle \rho, \tau \rangle$, and so $\text{Aut}(\Gamma) = \langle \rho, \tau \rangle$.

Let $\tau' \in \text{Aut}(\Gamma)$ as above. If $\sigma(i) = i$, then if $|\tau'| = q^{k-1}$, then as $|\langle \tau_1^q \rangle| = q^{k-1}$ and \mathbb{Z}_p^* is cyclic, $\tau' \in \langle \tau_1^q \rangle \leq \langle \tau_1 \rangle$. If $\tau'^q \geq q^k$, then as \mathbb{Z}_p^* is cyclic, the function $\delta : V \rightarrow V$ by $\delta(v_j^i) = v_{\alpha_j}^i$ is in $\text{Aut}(\Gamma)$, in which case Γ is circulant, a contradiction. If $\sigma(i) = i + r$, then $\tau_1^{-r}\tau'(v_j^i) = v_{\beta_{\alpha^{-r}j}}^i$, and by arguments above we must have that $|\alpha^{-r}\beta| \leq q^{k-1}$. Hence $\alpha^{-r}\beta \in \langle \alpha^q \rangle$ and so $\alpha^{-r}\beta = \alpha^s$. Then $\beta = \alpha^{r+s}$. If $\alpha^s = 1$, then $\beta = \alpha^r$ and so $\tau' = \tau^r$ and $\tau' \in \langle \tau_1 \rangle$. Further, observe that by the above arguments, we also have that $|\beta| = |\alpha|$, or Γ is circulant. Hence $|\beta| = |\alpha|$, and $r + s \not\equiv 0 \pmod{q}$. Also note that as $|\alpha^{-r}\beta| = q^{k-1}$, $s \equiv 0 \pmod{q}$, and so $r \not\equiv 0 \pmod{q}$. As the function $\delta_s : V \rightarrow V$ by $\delta_s(v_j^i) = v_{\alpha^s j}^i$ is in $\text{Aut}(\Gamma)$, $\tau' = \tau_1^r \gamma_s$ and $\gamma_s \in \langle \tau^q \rangle$. Then $\tau' \in \langle \tau \rangle$ and the result follows.

(ii) If $\text{Aut}(\Gamma)$ admits only a complete block system of p blocks of size q , then the map $\pi_1 : \text{Aut}(\Gamma) \rightarrow S_p$ is well defined. Further, if $\text{Ker}(\pi_1) \neq 1$, then Γ is a (p, q) -metacirculant and is thus circulant. Hence $\text{Ker}(\pi_1) = 1$. If $\text{Aut}(\Gamma)/q$ contains

a normal subgroup H , then H is transitive and so $p \mid |H|$. Also, $\pi_1^{-1}(H) \triangleleft \text{Aut}(\Gamma)$, and as the orbits of $\pi_1^{-1}(H)$ have order p , $\text{Aut}(\Gamma)$ admits a complete block system of q blocks of size p . Thus $\pi_1(\text{Aut}(\Gamma))$ is nonsolvable and doubly transitive.

It follows by [46] that if p and q are odd primes, then $p = (k^n - 1)/(k - 1)$ for some prime n and a prime power k such that $(n, k - 1) = 1$ and that the minimal transitive subgroup of $\text{Aut}(\Gamma)$ that admits a complete block system of p blocks of size q is isomorphic to $SL(2, 2^s)$. Further, by [46], Prop 2.4, $\text{Aut}(\Gamma)$ is isomorphic to a subgroup of $\text{Aut}(SL(2, 2^s))$. If $q = 2$, then by the proof of [46], Theorem, Γ is metacirculant if and only if Γ is the wreath product of an order p circulant over an order 2 circulant, which is isomorphic to a circulant graph. \square

Theorem 3.8. Let Γ be a vertex-transitive graph of order pq , $q < p$ primes. If $\text{Aut}(\Gamma) \cong G$ is primitive, then G is one of the groups in Table 1. If $\text{Aut}(\Gamma) \cong G$ is imprimitive, then there exist $A < N(p)$, $B < N(q)$, and Γ_1, Γ_2 , vertex-transitive graphs of order p or q such that one of the following is true:

(i) $G = S_q \times S_p$,

(ii) $G = A \times S_p$,

(iii) $G = S_q \times B$.

(iv) $G < N(pq)$,

(v) $G = \langle \rho, \tau \rangle$ for some choice of $\alpha \in \mathbb{Z}_p^*$,

(vi) $G = \text{Aut}(\Gamma_1) \wr \text{Aut}(\Gamma_2)$,

(vii) $p = 2^{2^s} + 1$ is a Fermat prime, and q divides $2^{2^s} - 1$. Further, the minimal transitive subgroup of $\text{Aut}(\Gamma)$ that admits only a complete block system of p blocks of size q is isomorphic to $SL(2, 2^s)$ and $\text{Aut}(\Gamma)$ is isomorphic to a subgroup of $\text{Aut}(SL(2, 2^s))$.

Proof. If $\text{Aut}(\Gamma)$ is imprimitive, the result follows from Theorems 3.6, and 3.7. If $\text{Aut}(\Gamma)$ is primitive, then the result follows from [46], [52], [57]. \square

Table 1.

Automorphism groups G of vertex primitive graphs of order pq .		
pq	G	$\text{soc}(G)$
pq	S_{pq}	A_{pq}
$p(p-1)/2$	S_p	A_p
$p(p+1)/2$	S_{p+1}	A_{p+1}
$(2^d \pm 1)(2^{d-1} \pm 1)$	$O^\pm(2d, 2)$	$\Omega^\pm(2d, 2)$
$(k+1)(k^2+1)$	$\text{P}\Gamma\text{S}_p(4, k)$	$\text{P}\text{Sp}(4, k)$
$q(q^2+1)/2$	$\text{P}\Sigma\text{L}(2, k^2)$	$\text{PSL}(2, k^2)$
$p(p \pm 1)/2$	$\text{PSL}(2, p)$	$\text{PSL}(2, p)$
$3 \cdot 7$	$\text{PGL}(2, 7)$	$\text{PGL}(2, 7)$
$5 \cdot 7$	S_7	A_7
$5 \cdot 7$	$\text{P}\Gamma\text{L}(4, 2)$	$\text{PSL}(4, 2)$
$5 \cdot 11$	$\text{PGL}(2, 11)$	$\text{PGL}(2, 11)$
$7 \cdot 11$	$\text{Aut}(M_{22})$	M_{22}
$5 \cdot 31$	$\text{P}\Gamma\text{L}(5, 2)$	$\text{PSL}(5, 2)$
$7 \cdot 29$	$\text{PSL}(2, 29)$	$\text{PSL}(2, 29)$
$11 \cdot 23$	$\text{PSL}(2, 23)$	$\text{PSL}(2, 23)$
$29 \cdot 59$	$\text{PSL}(2, 59)$	$\text{PSL}(2, 59)$
$31 \cdot 61$	$\text{PSL}(2, 61)$	$\text{PSL}(2, 61)$
$3 \cdot 19$	$\text{PSL}(2, 19)$	$\text{PSL}(2, 19)$

3.3 BURNSIDE GROUPS

Let G be a finite group. The *socle* of G , $\text{soc}(G)$ is the product of all minimal normal subgroups of G . If G acts transitively on Ω and G is primitive but not doubly transitive, we say G is *simply primitive*. Let G be a primitive group of degree pq , and $\Gamma_1, \Gamma_2, \dots, \Gamma_s$ the orbital graphs of G . Clearly $\text{Aut}(\Gamma_i)$, $1 \leq i \leq s$ is also primitive, as $G \leq \text{Aut}(\Gamma_i)$. Thus to determine which regular groups G of order pq are not Burnside groups, it suffices to check which vertex-transitive graphs of order pq that have primitive automorphism groups are also Cayley graphs. This was done by Praeger and Xu [52]. Note that in all cases, the regular group of order pq contained in a primitive group must be nonabelian.

Theorem 3.9. Let G be a nonabelian regular group of order pq . Then G is not a Burnside group if and only if pq is one of the values listed in Table 2.

Table 2.

Simply primitive groups G of degree pq that contain a regular subgroup.	
pq	$\text{soc}(G)$
$p(p-1)/2$	A_p
$p(p \pm 1)/2, p \equiv 3 \pmod{4}$	$\text{PSL}(2, p)$
$3 \cdot 7$	$\text{PGL}(2, 7)$
$5 \cdot 11$	$\text{PGL}(2, 11)$
$5 \cdot 31$	$\text{PSL}(5, 2)$
$7 \cdot 29$	$\text{PSL}(2, 29)$
$11 \cdot 23$	$\text{PSL}(2, 23)$
$29 \cdot 59$	$\text{PSL}(2, 59)$

We now wish to generalize the notion of a Burnside group to non-regular imprimitive permutation groups. Let G be a group of order n that acts transitively

but not doubly transitively on a set Ω of order $m < n$ and has no nontrivial transitive subgroups. We say that G is an m -Burnside group if whenever the minimal transitive subgroup of a group H acting on a set of size $|\Omega|$ is isomorphic to G , then H is either doubly transitive or imprimitive. Again to determine which groups G of degree pq are not pq -Burnside groups, we only need check which groups are minimal transitive subgroups of the minimally transitive primitive subgroups of automorphism groups of vertex-transitive graphs.

Theorem 3.10. Let $p > q$ be prime and G a group of degree pq that contains no proper transitive subgroups. Then G is not a pq -Burnside group if and only if G is one of the simple groups in Table 3 or $G = \langle \rho, \tau \rangle$ where ρ and τ are defined as in the introduction, $|\tau| = pq^2$, and $pq = 10$ or 57 .

Table 3.

Simply primitive groups G of degree pq that are minimally transitive.	
pq	$\text{soc}(G)$
$p(p+1)/2$	A_{p+1}
$(2^d \pm 1)(2^{d-1} \pm 1)$	$\Omega^\pm(2d, 2)$
$(k+1)(k^2+1)$	$\text{PSp}(4, k)$
$q(q^2+1)/2$	$\text{PSL}(2, k^2)$
$p(p \pm 1)/2, p \equiv 1 \pmod{4}$	$\text{PSL}(2, p)$
$5 \cdot 7$	A_7
$5 \cdot 7$	$\text{PSL}(4, 2)$
$7 \cdot 11$	M_{22}
$31 \cdot 61$	$\text{PSL}(2, 61)$

Proof. We need only determine the minimal transitive subgroups of the groups listed in Table 1 which are not in Table 2. Suppose G is as in Table 1 such that G does not contain a regular subgroup, and let H be a minimal transitive subgroup

of G . If H is imprimitive, then by [46], any graph that has G as an automorphism group is either circulant, metacirculant, or is contained in $\text{Aut}(SL(2, 2^s))$ for s, q, p as in Theorem 3.6. Note that $\text{Aut}(SL(2, 2^s))$ is itself imprimitive. One can easily check that, of the values of pq in Table 1 but not in Table 2, only $pq = 10$ and $pq = 57$ can possibly admit metacirculant graphs that are not Cayley. If $pq = 10$, then it is well known that (see [3]) the Petersen graph is metacirculant but not Cayley. Hence we only need consider when $pq = 57$ and $G = PSL(2, 19)$.

Now, $|PSL(2, 19)| = 19 \cdot 180$, so the Sylow 19-subgroups of $PSL(2, 19)$ have order 19 and there are 20 of them. Let P be a Sylow 19-subgroup of $PSL(2, 19)$. Then, acting on a set S of order 57, if $P = \langle \delta \rangle$, then δ is semiregular. Let $S = \{v_j^i : i \in \mathbb{Z}_3, j \in \mathbb{Z}_{19}\}$ and assume without loss of generality that $\delta(v_j^i) = v_{j+1}^i$. As $[PSL(2, 19) : N_{PSL(2, 19)}(P)] = 20$, we have that $|N_{PSL(2, 19)}(P)| = 9 \cdot 19$. If $N_{PSL(2, 19)}(P)$ is transitive, then by Corollary 3.3, every orbital graph of $PSL(2, 19)$ is a metacirculant graph. If $N_{PSL(2, 19)}(P)$ is not transitive, then $N_{PSL(2, 19)}(P) \cong \langle \rho, \gamma \rangle$, where $\gamma(v_j^i) = v_{9j}^i$. By [18], there exists an orbital graph Δ of $PSL(2, 19)$ such that Δ is regular of degree 6. However, $|\gamma| = 9$ so v_0^0 is adjacent to either 2 vertices of Δ or at least 9 vertices of Δ , a contradiction. \square

CHAPTER 4

ISOMORPHISM PROBLEMS FOR VERTEX-TRANSITIVE GRAPHS OF ORDER A PRODUCT OF TWO PRIMES

In [3], Alspach and Parsons posed the question of determining necessary and sufficient conditions for two (m, n) -metacirculant graphs to be isomorphic. We will solve this problem for (q, p) -metacirculant graphs. In order to solve this problem, we will extend Babai's characterization of the CI-property (Lemma 2.4) to vertex-transitive hypergraphs that are not Cayley. This extension of Lemma 2.4 will then allow us to classify vertex-transitive graphs of order pq using the characterization of vertex-transitive graphs given by Cheng and Oxley [18], Marušič and Scapellato [46], Prager and Xu [52], and Wang and Xu [57]. We then give an algorithm to explicitly calculate the isomorphism classes of (q, p) -metacirculant graphs.

Also in [3], Alspach and Parsons gave necessary and sufficient conditions for a (q, p, α) -metacirculant graph Γ such that $p^2 \mid |\alpha|$ to be a Cayley graph. We improve their conditions by showing that any such graph Γ is Cayley if and only if it is in fact circulant.

4.1 CHARACTERIZING CAYLEY METACIRCULANT GRAPHS

We first improve Alspach and Parsons characterization of (q, p) -metacirculant graphs that are Cayley graphs. We begin with necessary and sufficient conditions for a (q, p) -metacirculant combinatorial object to be a Cayley object when p^2 does not divide $|\text{Aut}(X)|$.

Theorem 4.1. Let $X = X(q, p, \alpha)$ be a metacirculant combinatorial object, $p > q$, such that $p^2 \nmid |\text{Aut}(X)|$. Then X is a Cayley object if and only if $X = X(q, p, \alpha')$ where $|\alpha'| = 1$ or $|\alpha'| = q$. Further, if $q^2 \mid |\alpha|$, then X is a Cayley object if and only if X is circulant.

Proof. By Theorem [3] it suffices to show necessity. Let $X = X(q, p, \alpha)$ satisfy the hypothesis and suppose that X is a Cayley object. As X is a Cayley object, X contains the left translations of some group of order pq , and hence contains a regular subgroup, say G . As the Sylow p -subgroups of $\text{Aut}(X)$ have order p , by conjugating G , if necessary, we may assume without loss of generality that $\langle \rho \rangle \leq G$. Further, $\langle \rho \rangle$ is also a Sylow p -subgroup of G and, as $|G| = pq$, $\langle \rho \rangle \triangleleft G$, and certainly $\langle \rho \rangle$ is not transitive. Hence G admits a complete block system of q blocks each of size p , where the blocks are formed by the orbits of $\langle \rho \rangle$. Note that we may assume that $G = \langle \rho, \tau_1 \rangle$ where $|\tau_1| = q$ as upto isomorphism there are exactly two groups of order pq , both of which can be generated in this fashion [34]. We conclude that $G = \langle \rho, \tau_1 \rangle$ where $\tau_1(v_j^i) = v_{\alpha_1 j + a_i}^{\sigma(i)}$, $\sigma \in S_q$, $\alpha_1 \in \mathbb{Z}_p^*$, and $a_i \in \mathbb{Z}_p$.

Let $N = N_{\text{Aut}(X)}(\langle \rho \rangle)$ the normalizer in $\text{Aut}(X)$ of $\langle \rho \rangle$. Clearly N admits a complete block system \mathcal{B} of q blocks of size p , where the blocks are orbits of $\langle \rho \rangle$. Define $\pi_1 : N \rightarrow S_q$ by $\pi_1(\gamma) = \gamma/\mathcal{B}$, and let $K = \text{Ker}(\pi_1)$. Note that $\tau, \tau_1 \in N$ where $\tau(v_j^i) = v_{\alpha j}^{i+1}$, $\alpha \in \mathbb{Z}_p^*$. If $q^2 \nmid |\alpha|$ we are done so assume that $q^2 \mid |\alpha|$. We must show that X is circulant. As $\tau^q(v_0^0) = v_0^0$ but $\tau^q \neq 1$, $\text{Stab}_K(v_0^0) \neq 1$. As $p^2 \nmid |\text{Aut}(X)|$, by a previous argument we have $\text{Ker}(\pi_2) = 1$, and hence the equivalence classes of $\equiv E_0, \dots, E_{p-1}$ each have cardinality q , where each E_i contains exactly

one element from each orbit of ρ . As τ^q only fixes $v_0^0, v_0^1, \dots, v_0^{q-1}$ and $\rho \in K$, we may take $E_i = V_i$. As $\rho \in K$, we have that $a_i = a_j$ for every $i, j \in \mathbb{Z}_q$. Hence $\rho^{-a_0}\tau_1(v_j^i) = v_{\alpha_1 j}^{\sigma(i)}$. Let $\tau_2 = \rho^{-a_0}\tau_1$.

Note that $\langle \tau_2 \rangle / p = \langle \sigma \rangle$ and $\langle \tau \rangle / p$ are Sylow q -subgroups of N/p . Thus there exists $\beta \in N$ such that $(\beta^{-1}/p)(\langle \tau_2 \rangle / p)(\beta/p) = \langle \tau \rangle / p$. Then $\beta^{-1}\tau_2\beta \in N$ and $\beta^{-1}\tau_2\beta(V^i) = V^{i+w}$, for some $w \in \mathbb{Z}_q$. We conclude that $\beta^{-1}\tau_2\beta(v_j^i) = v_{\alpha_2 j + b_i}^{i+w}$, for some α_2 and $b_i \in \mathbb{Z}_p$. Further, by an argument analogous to a previous argument, we have that $b_i = b_j$ for every $i, j \in \mathbb{Z}_q$. Hence we may assume $\beta^{-1}\tau_2\beta(v_j^i) = v_{\alpha_2 j}^{i+w}$. Let $t \in \mathbb{Z}_q$ such that $tw \equiv 1 \pmod q$. Then $\beta^{-1}\tau_2^t\beta(v_j^i) = v_{\alpha_2^t j}^{i+1}$. Let $\tau' = \beta^{-1}\tau_2^t\beta$ and $\alpha' = \alpha_2^t$. Then $\tau'(v_j^i) = v_{\alpha' j}^{i+1}$ and, as $|\tau_1| = q$ and $(t, q) = 1$, $|\tau'| = q$. Hence $|\alpha'| = 1$ or $|\alpha'| = q$. If $\alpha' = 1$, then X is circulant as required. If $|\alpha'| = q$, then, as \mathbb{Z}_p^* is cyclic and $\tau \in \text{Aut}(X)$, then function $\gamma : V \rightarrow V$ by $\gamma(v_j^i) = v_{\alpha' j}^i$ is in $\text{Aut}(X)$. Hence $\gamma^{-1}\tau' \in \text{Aut}(X)$, and $\gamma^{-1}\tau'(v_i^j) = v_j^{i+1}$. We conclude that X is circulant. \square

Corollary 4.2. Let $\Gamma = \Gamma(q, p, \alpha, S_0, \dots, S_\mu)$ where $p > q$. Then Γ is a Cayley graph if and only if $\Gamma = \Gamma(q, p, \alpha', S_0, \dots, S_\mu)$, $|\alpha'| = q$ or $|\alpha'| = 1$. Further, if $q^2 \mid |\alpha|$, then Γ is a Cayley graph if and only if Γ is circulant.

Proof. If the Sylow p -subgroups of $\text{Aut}(\Gamma)$ have order p , then the result follows from Theorem 4.1. If the Sylow p -subgroups of $\text{Aut}(\Gamma)$ have order greater than p , then by Lemma 3.1, they have order p^{q+1} , and as $\rho \in \text{Aut}(\Gamma)$, $\Gamma = \Gamma_1 \wr \Gamma_2$, where Γ_1 is an order q -circulant and Γ_2 an order p -circulant. If $\Gamma = K^{pq}$ or E^{pq} , the

result is trivial, and if $E^{pq} \neq \Gamma \neq K^{pq}$, $\text{Aut}(\Gamma) = \text{Aut}(\Gamma_1) \wr \text{Aut}(\Gamma_2)$. Hence Γ is circulant. \square

We illustrate this corollary with an example.

Example 4.3. The Petersen graph is not Cayley.

Proof. By [3], the Petersen graph is a $(2, 5, 2, \{1, 4\}, \{0\})$ metacirculant graph. Clearly, the Petersen graph satisfies the hypothesis of Corollary 4.2, and so is Cayley if and only if it is circulant. By inspection, the Petersen graph is not circulant with this labeling, and is thus not Cayley. \square

4.2 ISOMORPHISM PROBLEM FOR CAYLEY METACIRCULANT GRAPHS

We now turn our attention to the isomorphism problem for (q, p) -metacirculant graphs that are Cayley graphs. Let $p > q$, $q|p-1$, and $\alpha \in \mathbb{Z}_p^*$ such that $|\alpha| = q$. Then upto isomorphism, $\langle \rho, \tau \rangle$ is one of two groups of order pq , the other being \mathbb{Z}_{pq} .

Theorem 4.4. Let X be a Cayley object of G with $|G| = pq$, $p > q$, such that $p^2 \nmid |\text{Aut}(X)|$. If $G = \mathbb{Z}_{pq}$ then X is a CI-object for G . If $G = \langle \rho, \tau \rangle$, for some α as above, then either X is a CI-object for G or X is also a Cayley object for \mathbb{Z}_{pq} .

Proof. If $G = \mathbb{Z}_{pq}$, then the result follows from the proof of Theorem 1, Case 1 [2].

If $G = \langle \rho, \tau \rangle$, for some α , we will show that if $\phi \in S_{pq}$ such that $\phi^{-1}\langle \rho, \tau \rangle\phi \leq \text{Aut}(X)$, then either $\phi^{-1}\langle \rho, \tau \rangle\phi$ and $\langle \rho, \tau \rangle$ are conjugate in $\text{Aut}(X)$, or that X

is also a Cayley object for \mathbb{Z}_{pq} . For brevity, let $\phi_1 = \phi^{-1}\tau\phi$. By hypothesis, $\langle \rho \rangle$ and $\langle \phi^{-1}\rho\phi \rangle$ are Sylow p -subgroups of $\text{Aut}(X)$ and are thus conjugate. Let $\delta_1 \in \text{Aut}(X)$ such that $\delta_1^{-1}\langle \phi^{-1}\rho\phi \rangle\delta_1 = \langle \rho \rangle$. Thus $\delta_1^{-1}\langle \phi^{-1}\rho\phi, \phi_1 \rangle\delta_1 = \langle \rho, \delta_1^{-1}\phi_1\delta_1 \rangle$. Let $\phi_2 = \delta_1^{-1}\phi_1\delta_1$. Clearly $\langle \rho \rangle \triangleleft \langle \rho, \phi_2 \rangle$ and so $\langle \rho, \tau, \phi_2 \rangle$ admits a complete block system \mathcal{B} of q blocks each of size p . Define $\pi_1 : \langle \rho, \tau, \phi_2 \rangle \rightarrow S_q$ by $\pi_1(\gamma) = \gamma/\mathcal{B}$. Hence $\langle \tau \rangle/p$ and $\langle \phi_2 \rangle/p$ are Sylow q -subgroups of $\langle \rho, \tau, \phi_2 \rangle/p$. Let $\delta_2 \in \langle \rho, \tau, \phi_2 \rangle$ such that $(\delta_2^{-1}/p)(\langle \phi_2 \rangle/p)(\delta_2/p) = \langle \tau \rangle/p$. Let $\phi_3 = \delta_2^{-1}\phi_2\delta_2$. Let Q be a Sylow q -subgroup of $\langle \rho, \tau, \phi_3 \rangle$. If $|Q| = q$ then $\langle \tau \rangle$ and $\langle \phi_2 \rangle$ are Sylow q -subgroups of $\langle \rho, \tau, \phi_3 \rangle$ and so there exists $\delta_3 \in \langle \rho, \tau, \phi_2 \rangle$ such that $\delta_3^{-1}\langle \phi_2 \rangle\delta_3 = \langle \tau \rangle$. Let $\delta = \delta_3\delta_2\delta_1$. Then $(\delta\phi)^{-1}\langle \rho, \tau \rangle\delta\phi = \langle \rho, \tau \rangle$. Thus by Lemma 2.4, if $|Q| = q$, then X is a CI-object for $\langle \rho, \tau \rangle$.

If $|Q| = q^a$, $a > 1$, we will show that X is a Cayley object for \mathbb{Z}_{pq} . As $\langle \phi_3 \rangle/p = \langle \tau \rangle/p$, there exist $\beta \in \text{Ker}(\pi_1)$ such that $|\beta| = q^b$, $b \geq 1$. Without loss of generality we may assume that $b = 1$. As $\langle \rho \rangle \triangleleft \langle \rho, \tau, \phi_2 \rangle$, $\beta(v_j^i) = v_{\alpha_i, j+b_i}^i$ where $\alpha_i \in \mathbb{Z}_p^*$ and $b_i \in \mathbb{Z}_p$. By a previous argument, as $p^2 \nmid |\text{Aut}(X)|$, $\alpha_i = \alpha_0$ for all i , and as $|\beta| = q$, we must have $|\alpha_0| = q$. Now, \mathbb{Z}_p^* is cyclic of order $p-1$ so there exists $r \in \mathbb{Z}_p^*$ such that $\alpha_0^r = \alpha^{-1}$. Hence $\beta^r\tau \in \langle \rho, \tau, \phi_2 \rangle$ and $\beta^r\tau(v_j^i) = v_{j+c_i}^{i+1}$, where each $c_i \in \mathbb{Z}_p$.

Let $K = \text{Ker}(\pi_1)$. As the Sylow p -subgroups of $\text{Aut}(X)$ have order p , by a previous argument $\text{Ker}(\pi_2) = 1$. As $\beta \in K$, $\text{Stab}_K(v_0^0) \neq 1$ and, again by a previous argument, we have that the equivalence classes E_0, E_1, \dots, E_{p-1} of \equiv have cardinality q . We must then have that $\langle \rho, \tau, \phi_2 \rangle$ admits a complete block system of p

blocks each of size q , where the blocks are the equivalence classes E_0, E_1, \dots, E_{p-1} .

As $\tau \in \langle \rho, \tau, \phi_2 \rangle$, we conclude that $c_i = c_j$ for all i, j . Hence X is a Cayley object for \mathbb{Z}_{pq} . \square

Corollary 4.5. Let $X = X(p, q, \alpha)$ and $X' = X'(p, q, \alpha')$ be metacirculant combinatorial objects such that X and X' are Cayley objects and $p^2 \nmid |\text{Aut}(X)|$. Then X is isomorphic to X' if and only if (i) if X is circulant then there exists $\delta \in \text{Aut}(\mathbb{Z}_{pq})$ such that $\delta(X) = X'$,

(ii) if X is not circulant then there exists $\delta \in \text{Aut}(\langle \rho, \tau \rangle)$ and $\gamma : V \rightarrow V$ where $\gamma(v_j^i) = v_j^{r_i}$, $r \in \mathbb{Z}_q^*$ and $\gamma\delta(X) = X'$.

Further if X and X' are isomorphic, then X is circulant if and only if X' is circulant.

Proof. (i) In view of Theorem 4.4, if $a = |\alpha|$ and $a' = |\alpha'|$, we may assume without loss of generality that $a = 1$ and $a' = 1$ or $a' = q$. If $a' = 1$, then X and X' are circulant and the result follows from Theorem 4.4. Hence we assume that $a' = q$, i.e. that X is circulant and X' is not necessarily circulant. We will show that if X and X' are isomorphic, then X' is circulant implying (i) and by symmetry, that X is circulant if and only if X' is circulant.

Assume $X \cong X'$. Then $\text{Aut}(X) \cong \text{Aut}(X')$ and so $\text{Aut}(X')$ contains a pq cycle ω_0 . As $p^2 \nmid |\text{Aut}(X)|$, $\langle \omega_0^q \rangle$ is a Sylow p -subgroup of $\text{Aut}(X')$ and $N_{\text{Aut}(X')}(\langle \omega_0^q \rangle)$ contains the pq cycle ω_0 . Further $\langle \rho \rangle$ is also a Sylow p -subgroup of $\text{Aut}(X')$ and so there exist $\beta_0 \in \text{Aut}(X')$ such that $\beta_0^{-1} \langle \omega_0^q \rangle \beta_0 = \langle \rho \rangle$ and $\beta_0^{-1} N_{\text{Aut}(X')}(\langle \omega_0^q \rangle) \beta_0^{-1} =$

$N_{\text{Aut}(X')}(\langle \rho \rangle)$. Let $\omega_1 = \beta_0^{-1} \omega_0 \beta_0$ and $R = N_{\text{Aut}(X')}(\langle \rho \rangle)$. Then $\langle \omega_1 \rangle \leq R$, $\langle \omega_1^q \rangle = \langle \rho \rangle$, and $\langle \omega_1 \rangle$ is cyclic of order pq . As $\langle \rho \rangle \triangleleft R$, R admits a complete block system \mathcal{B} of q blocks each of size p , where the blocks are formed by the orbits of ρ . Define $\pi_1 : R \rightarrow S_q$ by $\pi_1(\gamma) = \gamma/\mathcal{B}$. Then $\langle \tau \rangle/p$ and $\langle \omega_1^p \rangle/p$ are Sylow q -subgroups of R/p . Thus there exist $\beta_1 \in R$ such that $(\beta_1^{-1}/p)(\langle \omega_1^p \rangle/p)(\beta_1/p) = \langle \tau \rangle/p$. Let $\omega = \beta_1^{-1} \omega_1 \beta_1$. Let V_i be a block of size p that contains v_0^i . As $\langle \omega^p \rangle/p = \langle \tau \rangle/p$, $\omega(V_i) = V_{i+w}$ for some $w \in \mathbb{Z}_q$. As $\langle \rho \rangle \triangleleft \langle \omega \rangle$, $\omega(v_j^i) = v_{\phi j + b_i}^{i+w}$, where $\phi \in \mathbb{Z}_p^*$, and $b_i \in \mathbb{Z}_p$. Trivially, either the Sylow q -subgroups of R are either of order q or of order q^i , $i > 1$. In either case, the result follows with arguments analogous to those in Theorem 4.4.

(ii) If $p^2 \nmid |\text{Aut}(X)|$ and X is not circulant, then X is a Cayley graph for $\langle \rho, \tau \rangle$. Further, there exists $r \in \mathbb{Z}_q$ such that $\gamma^{-1}(X')$ is also a Cayley graph for $\langle \rho, \tau \rangle$. Hence by Lemma 2.4, X and $\gamma^{-1}(X')$ are isomorphic if and only if there exists $\delta \in \text{Aut}(\langle \rho, \tau \rangle)$ such that $\delta(X) = \gamma^{-1}(X')$. Hence X and X' are isomorphic if and only if $\gamma\delta(X) = X'$. \square

Corollary 4.6. Let $\Gamma = \Gamma(p, q, \alpha, S_0, \dots, S_\mu)$ and $\Gamma' = \Gamma'(p, q, \alpha', S'_0, \dots, S'_\mu)$ be metacirculant graphs that are Cayley graphs. Then Γ is isomorphic to Γ' if and only if

(i) if Γ is circulant then there exists $\delta \in \text{Aut}(\mathbb{Z}_{pq})$ such that $\delta(\Gamma) = \Gamma'$,

(ii) if Γ is not circulant then there exists $\delta \in \text{Aut}(\langle \rho, \tau \rangle)$ and $\gamma : V \rightarrow V$ where $\gamma(v_j^i) = v_j^{ri}$, $r \in \mathbb{Z}_q^*$, and $\gamma\delta(\Gamma) = \Gamma'$.

Further, if Γ and Γ' are isomorphic then Γ is circulant if and only if Γ' is circulant.

Proof. If p^2 does not divide $|\text{Aut}(\Gamma)|$, the result follows from Corollary 4.5. If $p^2 \mid |\text{Aut}(\Gamma)|$, then it follows from [2] and Lemma 3.10. \square

4.3 EXTENDING THE CI-PROPERTY TO NON-CAYLEY GRAPHS

We now consider the isomorphism problem for (q, p) -metacirculant graphs that are not Cayley graphs. Let X and X' be (q, p) -metacirculant combinatorial objects that are not Cayley. Initially, determining necessary and sufficient conditions for X and X' to be isomorphic is hampered by no result corresponding to Babai's characterization of the CI-property for Cayley objects. Sabidussi [55] proved that some 'multiple' $n\Gamma$ of Γ is a Cayley graph. We first generalize Sabidussi's result to vertex-transitive hypergraphs, and then use Babai's characterization of the CI-property for Cayley objects to characterize an analogous isomorphism result for non-Cayley hypergraphs.

Lemma 4.7. A combinatorial object X is isomorphic to a Cayley object of G if and only if $\text{Aut}(X)$ contains a regular subgroup isomorphic to G .

Proof. If X is a Cayley object of G then G_L is a regular subgroup of $\text{Aut}(\Gamma)$. If $\text{Aut}(X)$ contains a regular subgroup S isomorphic to G , then by Schur's method

[58] we may relabel $V(X)$ with elements of S so that $S_L \leq \text{Aut}(X)$. Hence X is isomorphic to a Cayley object of G . \square

Given a vertex-transitive hypergraph X , we say that X is *reducible* if there exists a vertex-transitive hypergraph Y and an integer $n > 1$ such that $X \cong Y \wr E^n$. Otherwise, X will be said to be *irreducible*. Before generalizing Sabidussi's result, we need to prove the following technical lemma.

Lemma 4.8. Let X be a reducible vertex-transitive hypergraph, Y an irreducible vertex-transitive hypergraph and n an integer such that $X = Y \wr E^n$. Then $\text{Aut}(X) = \text{Aut}(Y) \wr S_n$, and the orbits of $1 \wr S_n$ form a complete block system for $\text{Aut}(X)$.

Proof. Note that $\text{Aut}(Y) \wr S_n \leq \text{Aut}(X)$, and the orbits of $1 \wr S_n$ form a complete block system for $\text{Aut}(Y) \wr S_n$. Denote the blocks of size n by B_0, B_1, \dots, B_k . We note that it suffices to show that B_0, \dots, B_k are blocks of $\text{Aut}(X)$. Assume not. Then there exists $\alpha \in \text{Aut}(X)$ such that $\alpha(B_i) \cap B_i \neq \emptyset$ and $\alpha(B_i) \neq B_i$, for some $0 \leq i \leq k$. Then there exists $x \in B_i$ and $y \notin B_i$ such that $\alpha(x) \in B_i$ and $x' \in B_i$ such that $\alpha(x') = y$. If there exists an edge $(x_1, \dots, x_r) \in E(X)$ such that $x = x_a, y = x_b$ for some a, b then $\alpha^{-1}(x_1, \dots, x_r) \in E(X)$, contradicting the fact that $X = Y \wr E^n$. Hence we assume that no such edge exists. Let $s = \max\{r : (y_1, y_2, \dots, y_r) \in E(Y)\}$. Define \tilde{Y} to be the hypergraph with vertex set $V(Y)$ and

$$E(\tilde{Y}) = \{(y_1, \dots, y_r) : 2 \leq r \leq s \text{ and } (y_1, \dots, y_r) \notin E(Y)\}.$$

Let $\tilde{X} = \tilde{Y} \wr E^n$. We will show that $\text{Aut}(X) = \text{Aut}(\tilde{X})$. Note that this will imply the result as there exists $e = (x_1, x_2, \dots, x_r) \in E(\tilde{X})$ such that $x_a = x$ and $x_b = y$.

Let $\beta \in \text{Aut}(\tilde{X})$. If $e = (x_1, x_2, \dots, x_r) \in E(X)$, then $\beta(e) \notin E(\tilde{X})$. Hence $\beta(e) \in E(X)$. Thus $\beta \in \text{Aut}(X)$ and so $\text{Aut}(\tilde{X}) \leq \text{Aut}(X)$. Conversely, let $\beta \in \text{Aut}(X)$ and e be an edge of \tilde{X} . Then $\beta(e) \notin E(X)$ so $\beta(e) \in E(\tilde{X})$. Thus $\text{Aut}(X) = \text{Aut}(\tilde{X})$. \square

If X, Y , and n satisfy the hypothesis of Lemma 4.8, then Y will be denoted by X_* and B_0, \dots, B_k will be denoted by x_*, y_* , etc., where $x \in B_i$, $y \in B_j$, etc. Observe that Lemma 6.14 implies that if X and X' are isomorphic vertex-transitive hypergraphs and $\delta : X \rightarrow X'$ is an isomorphism, then $\delta_* : X_* \rightarrow X'_*$ is an isomorphism where $\delta_*(x_*) = y_*$ if and only if $\delta(x) \in y_*$. Further $n\delta : V(X \wr E^n) \rightarrow V(X' \wr E^n)$ where $n\delta((x, a)) = (\delta(x), a)$ is also an isomorphism. Finally, if X and X' are irreducible, then $(n\delta)_* = \delta$.

Theorem 4.9. Let X be an irreducible vertex-transitive hypergraph, and $G \leq \text{Aut}(X)$ be transitive. Let $n = |\text{Stab}_G(x)|$, $x \in V(G)$. Then $X \wr E^n$ is a Cayley hypergraph for G .

Proof. By Lemma 4.7 it suffices to show that $\text{Aut}(X \wr E^n)$ contains a regular subgroup isomorphic to G . Clearly $\text{Aut}(X) \wr S_n \leq \text{Aut}(X \wr E^n)$. We will show that $G \wr S_n$ contains a regular subgroup isomorphic to G . For the moment, assume that X is a graph. Then by [55], Theorems 4,7, the result is true and $\text{Aut}(X \wr E^n) =$

$\text{Aut}(X) \wr S_n$. Observe that $\text{Aut}(X) \wr S_n$ admits a complete block system \mathcal{B} of $|X|$ blocks of size n , where the blocks are formed by the orbits of $1 \wr S_n$, and hence the map $\pi_1 : \text{Aut}(X) \wr S_n \rightarrow \text{Aut}(X)$, where $\pi_1(\gamma) = \gamma/\mathcal{B}$, is surjective. By [55], Theorem 2, we may label $V(X \wr E^n)$ with elements of G so that the blocks of size n are the left cosets in G of $\text{Stab}_G(x)$, for fixed $x \in V(X)$, and the vertices of X may be labeled with left cosets in G of $\text{Stab}_G(x)$. With this labeling, $\pi_1(g_L) = g$ for all $g \in G$. Hence $G_L \leq \pi_1^{-1}(G) = G \wr S_n$. Now, let X be an arbitrary irreducible vertex-transitive hypergraph and $n = |\text{Stab}_G(x)|$, for some $x \in V(X)$. By Lemma 4.8, $\text{Aut}(X \wr E^n) = \text{Aut}(X) \wr S_n \geq G \wr S_n$. As $\text{Aut}(K^r) = S_r$, the result follows. \square

Let X be a vertex-transitive hypergraph, and G a transitive subgroup of $\text{Aut}(X)$. Let $n = |\text{Stab}_G(x)|$, $x \in V(X)$. Then $X \wr E^n$ is a Cayley hypergraph of G . We will refer to X as an *n-Cayley hypergraph of G* . Assume that X is irreducible, and that if X' is another *n-Cayley hypergraph of G* then X and X' are isomorphic by α_* , $\alpha \in \text{Aut}(G)$. We then say that X is a *n-CI-hypergraph of G* .

Corollary 4.10. The following are equivalent:

- (i) X is an *n-CI-hypergraph of G* ,
- (ii) given a permutation $\phi \in S_{G_*}$, whenever $\phi^{-1}G\phi \leq \text{Aut}(X)$, then $\phi^{-1}G\phi$ and G are conjugate in $\text{Aut}(X)$.

Proof. Let X and X' be irreducible vertex-transitive hypergraphs such that $X \cong X'$ and X and X' are *n-Cayley hypergraphs of G* . As X and X' are irreducible, by

Lemma 4.8, if n is any integer then $\text{Aut}(X \wr E^n) = \text{Aut}(X) \wr S_n$. Let $n = |\text{Stab}_G(x_0)|$, $x_0 \in V(X) = V(X')$. By Theorem 4.9, $X \wr E^n$ and $X' \wr E^n$ are both Cayley hypergraphs for G and so if $X \wr E^n$ and $X' \wr E^n$ are isomorphic, then by Lemma 2.4 they are isomorphic by $\alpha \in \text{Aut}(G)$ if and only if whenever $\delta^{-1}G\delta \leq \text{Aut}(X \wr E^n)$ then $\delta^{-1}G\delta$ and G are conjugate in $\text{Aut}(X \wr E^n)$. Observe that as X and X' are irreducible, that if $(x, a) \in V(X \wr E^n) = V(X' \wr E^n)$, then $(x, a)_* = \{x\} \times N$, where N is a set of cardinality n , and that by Lemma 4.8, these sets together form a complete block system of $\text{Aut}(X \wr E^n)$ and $\text{Aut}(X' \wr E^n)$.

Assume that whenever $\delta^{-1}G\delta \leq \text{Aut}(X \wr E^n)$ then $\delta^{-1}G\delta$ and G are conjugate in $\text{Aut}(X \wr E^n)$. Let $\delta' \in S_V$ such that $\delta'^{-1}G\delta' \leq \text{Aut}(X)$. Then $(n\delta'^{-1})G(n\delta') \leq \text{Aut}(X \wr E^n)$ and so $(n\delta'^{-1})G(n\delta')$ and G are conjugate in $\text{Aut}(X \wr E^n)$. Then $(n\delta')_* = \delta'$. We conclude that $\delta'^{-1}G\delta'$ and G are conjugate in $\text{Aut}(X)$.

Now assume that whenever $\delta^{-1}G\delta \leq \text{Aut}(X)$ then $\delta^{-1}G\delta$ and G are conjugate in $\text{Aut}(X)$. Let $\delta' \in S_G$ such that $\delta'^{-1}G\delta' \leq \text{Aut}(X \wr E^n)$. By reversing the argument above we conclude that $\delta'^{-1}G\delta'$ is conjugate to $\beta^{-1}G\beta$ where $\beta_* = 1$. Hence $\beta^{-1} \in 1 \wr S_N$, $\beta \in \text{Aut}(X \wr E^n)$ and so $\beta\beta^{-1}G\beta\beta^{-1} = G$. We conclude that $\delta'^{-1}G\delta'$ is conjugate to G in $\text{Aut}(X \wr E^n)$. Thus if $\phi \in S_{G_*}$ such that $\phi^{-1}G\phi \leq \text{Aut}(X)$, then $\phi^{-1}G\phi$ and G are conjugate if and only if whenever $\phi' \in S_G$ and $\phi'^{-1}G\phi' \leq \text{Aut}(X \wr E^n)$, then $\phi'^{-1}G\phi'$ and G are conjugate in $\text{Aut}(X \wr E^n)$. Thus the result follows by Lemma 2.4. \square

4.4 ISOMORPHISM PROBLEM FOR NON-CAYLEY GRAPHS

We now apply Corollary 4.10 to determine the isomorphism classes of non-Cayley vertex-transitive graphs of order pq .

Corollary 4.11. Let $\text{SL}(2, 2^s)$ act imprimitively on \mathbb{Z}_{pq} , $p = 2^{2^s} + 1$ a Fermat prime, and $q|2^{2^s}$. Let $n = \text{Stab}_{\text{SL}(2, 2^s)}(0)$. Then $\text{SL}(2, 2^s)$ is an n -CI-group with respect to graphs.

Proof. Let Γ be a vertex-transitive graph of order pq with $\text{SL}(2, 2^s) \leq \text{Aut}(\Gamma)$. By Theorem 3.7, $\text{SL}(2, 2^s) \leq \text{Aut}(\Gamma) \leq \text{Aut}(\text{SL}(2, 2^s))$. Let $\varphi \in S_{pq}$ such that $\varphi^{-1}G_L\varphi \leq \text{Aut}(\Gamma)$. Then $\varphi^{-1}\text{SL}(2, 2^s)\varphi \leq \text{Aut}(\text{SL}(2, 2^s))$. Hence we have that $\varphi \in \text{Aut}(\text{SL}(2, 2^s))$ and so by Corollary 4.10, Γ is an n -CI-group with respect to graphs. □

Define the *deviation* of a vertex-transitive hypergraph X , $\text{dev}(X)$, to be the smallest integer n such that $X \wr E^n$ is a Cayley hypergraph.

We now restrict our attention to (q, p) -metacirculant hypergraphs X such that $q < p$ and X is not Cayley.

Lemma 4.12. If p^2 does not divide $|\text{Aut}(X)|$, then $\text{dev}(X) = q^{i-1}$, where q^i is the smallest power of q that divides $|\alpha|$, for any choice of α such that $\tau \in \text{Aut}(X)$.

Proof. It suffices to show that if $H \leq \text{Aut}(\Gamma)$ and H is transitive, then there exists H' such that $|H'| \leq H$ and $\langle \rho, \tau \rangle \leq |H'|$ for some choice of α . This follows with a proof similar to that of Theorem 4.1. \square

We remark that if Γ is a (q, p) -metacirculant graph that is not Cayley, then $p^2 \nmid |\text{Aut}(\Gamma)|$.

Theorem 4.13. Let $X = X(q, p, \alpha)$ be an irreducible metacirculant hypergraph that is not a Cayley hypergraph such that $p^2 \nmid |\text{Aut}(X)|$. Assume without loss of generality that $|\alpha| = q^k$. Then X is an q^{k-1} -CI-hypergraph.

Proof. We will show that whenever $\beta^{-1}\langle \rho, \tau \rangle \beta \leq \text{Aut}(\gamma(X))$ then $\beta^{-1}\langle \rho, \tau \rangle \beta$ and $\langle \rho, \tau \rangle$ are conjugate in $\text{Aut}(X)$. Let $\beta \in S_V$ such that $\beta^{-1}\langle \rho, \tau \rangle \beta \leq \text{Aut}(X)$. By arguments similar to those in Theorem 4.4, there exists $\sigma \in \text{Aut}(X)$ such that $(\sigma\beta)^{-1}\langle \rho, \tau \rangle (\sigma\beta) = \langle \rho, \tau' \rangle$ where $\tau'(v_j^i) = v_{\alpha'^i j}^{i+c}$, $c \in \mathbb{Z}_q^*$, $\alpha' \in \mathbb{Z}_p^*$ and $|\alpha'| = |\alpha|$. Let $N = N_{\text{Aut}(X)}(\langle \rho \rangle)$. Clearly $\tau, \tau' \in N$. If the Sylow q -subgroups of N have cardinality $q^{\text{dev}(X)+1}$ then $\langle \tau \rangle$ and $\langle \tau' \rangle$ are Sylow q -subgroups of N and are thus conjugate in N . Hence $\beta^{-1}\langle \rho, \tau \rangle \beta$ and $\langle \rho, \tau \rangle$ are conjugate in $\text{Aut}(X)$. Hence we assume that the Sylow q -subgroups of N have cardinality at least $q^{\text{dev}(X)+2}$. Let $\pi_1 : N \rightarrow S_q$ by $\pi_1(\gamma) = \gamma/\mathcal{B}$, where \mathcal{B} is the complete block system of N of q blocks of size p . Then there exist $\psi \in \text{Ker}(\pi_1)$ such that $|\psi| = q^{\text{dev}(X)+1}$. Hence by an argument similar to an argument in Theorem 4.1, we conclude that X is circulant and so X is Cayley, a contradiction. \square

Corollary 4.14. Let $X = X(q, p, \alpha)$ and $X' = X'(q, p, \alpha')$ be irreducible metacirculant hypergraphs that are not Cayley hypergraphs such that p^2 does not divide $|\text{Aut}(X)|$ or $|\text{Aut}(X')|$. Then X and X' are isomorphic if and only if there exist $\delta \in \text{Aut}(\langle \rho, \tau_1 \rangle)$, $\tau_1(v_j^i) = v_{\alpha_1 j}^{i+1}$, $|\alpha_1| = \text{dev}(X)/q = \text{dev}(X')/q$, $\gamma : V \rightarrow V$ where $\gamma(v_j^i) = v_j^{r_i}$, $r \in \mathbb{Z}_q^*$ and $\delta_* \gamma(X) = X'$.

Proof. It follows from arguments similar to arguments in Corollary 4.5 that there exists $\gamma \in S_V$, $\gamma(v_j^i) = v_j^{r_i}$, $r \in \mathbb{Z}_q^*$ such that if $n = \text{dev}(X)$ then $X \wr E^n$ and $X' \wr E^n$ are both Cayley hypergraphs for the group $\langle \rho, \tau_1 \rangle$, where $\tau_1(v_j^i) = v_{\alpha_1 j}^{i+1}$. Hence by Theorem 4.13 $\gamma(X)$ and X' are isomorphic by δ_* , $\delta \in \text{Aut}(\langle \rho, \tau_1 \rangle)$. \square

Corollary 4.15. Let $\Gamma = \Gamma(q, p, \alpha, S_0, \dots, S_\mu)$ and $\Gamma' = \Gamma'(q, p, \alpha', S'_0, \dots, S'_\mu)$ be metacirculant graphs that are not Cayley graphs. Then Γ and Γ' are isomorphic if and only if there exists $\delta \in \text{Aut}(\langle \rho, \tau_1 \rangle)$, $\tau_1(v_j^i) = v_{\alpha_1 j}^{i+1}$, $|\alpha_1| = \text{dev}(\Gamma)/q = \text{dev}(\Gamma')/q$, $\gamma : V \rightarrow V$ where $\gamma(v_j^i) = v_j^{r_i}$, $r \in \mathbb{Z}_q^*$ and $\delta_* \gamma(\Gamma) = \Gamma'$.

Proof. If $p^2 \mid |\text{Aut}(\Gamma)|$, or $p^2 \mid |\text{Aut}(\Gamma')|$, then Γ or Γ' is a Cayley graph. Hence the result follows from Corollary 4.14. \square

Note that Corollary 4.15 completes the classification of vertex-transitive graphs of order pq . Recall that if Γ is a vertex-transitive graph of order pq then Γ is circulant, metacirculant, or $\text{Aut}(\Gamma)$ has a simple minimal transitive subgroup. To classify vertex-transitive graphs of order pq , we need only determine when two graphs in

each of the above classes is isomorphic. If Γ is circulant then the classification follows from Theorem 3.11. If Γ is metacirculant, the classification follows from Corollary 4.6 and Corollary 4.15. If $\text{Aut}(\Gamma)$ contains a simple minimal transitive subgroup, then the classification follows from Corollary 4.11, or the minimal simple transitive subgroup of $\text{Aut}(\Gamma)$ is primitive. If the minimal simple transitive subgroup of $\text{Aut}(\Gamma)$ is primitive, then all such graphs are explicitly known (see [18], [52], and [57]).

In [4], Babai proved that if $|G| = 2p$, then G is a CI-group with respect to graphs. It would seem natural to ask for what values of $q|p - 1$, is $\langle \rho, \tau \rangle$ a q^k -CI-group, for $k \geq 0$.

Theorem 4.16. Let $q|p - 1$, $\alpha \in \mathbb{Z}_p^*$ such that $|\alpha| = q^{k+1}$, and $\tau(v_j^i) = v_{\alpha j}^{i+1}$. Then $\langle \rho, \tau \rangle$ is a weak q^k -CI-group via \mathbb{Z}_{pq} with respect to graphs, and is a q^k -CI-group with respect to graphs if and only if $q \leq 3$.

Proof. Let Γ be an q^k -Cayley graph for $\langle \rho, \tau \rangle$. By Theorem 4.13 and Corollary 4.15, $\langle \rho, \tau \rangle$ is a weak q^k -CI-group via \mathbb{Z}_{pq} with respect to graphs. Hence we need only show that $\langle \rho, \tau \rangle$ is a q^k -CI-group with respect to graphs if and only if $q \leq 3$. As $\langle \rho, \tau \rangle$ is a weak q^k -CI-group via \mathbb{Z}_{pq} with respect to graphs, we need only consider the case when Γ is also a Cayley graph for \mathbb{Z}_{pq} . Define $\tau_1 : V \rightarrow V$ by $\tau_1(v_j^i) = v_j^{i+1}$. Hence $\tau_1 \in \text{Aut}(\Gamma)$.

If $q \geq 3$, let $\alpha' \in \mathbb{Z}_p^*$ such that $|\alpha'| = q^{k+1}$. Define $\tau' : V \rightarrow V$ by $\tau'(v_j^i) = v_{\alpha' j}^{i+1}$. Denote the orbits of $\tau'^{-1}\tau_1$ of length q^{k+1} by $\mathcal{O}_1, \mathcal{O}_2, \dots, \mathcal{O}_s$. Let $1 \leq t \leq s$ such

that $\mathcal{O}_t \subseteq V^1$. Let $T = \mathcal{O}_t \cup \{v_j^{-i} : v_j^i \in \mathcal{O}_t\}$. Define a metacirculant graph Γ by $E(\Gamma) = \{v_b^a v_d^c : v_c^{-c} \in T\}$. Then Γ is circulant, $\tau' \in \text{Aut}(\Gamma)$, and, by Theorem 3.7 $\text{Aut}(\Gamma) = \langle \tau_1, \rho, \tau' \tau, \iota \rangle$. Let $\tau_2 : V \rightarrow V$ By $\tau_2(v_j^i) = v_{\alpha_j^i}^{i+1}$. Then $\tau_2 \in \text{Aut}(\Gamma)$, $\langle \rho, \tau_2 \rangle \cong \langle \rho, \tau \rangle$, and it is not difficult to see that $\langle \rho, \tau_2 \rangle$ is not conjugate to $\langle \rho, \tau \rangle$ in $\text{Aut}(\Gamma)$. Hence Γ is not a q^k -CI-graph and so $\langle \rho, \tau \rangle$ is not a q^k -CI-group with respect to graphs.

Let Γ' be a q^k -Cayley graph for $\langle \rho, \tau \rangle$ such that Γ' is isomorphic to Γ , and $\varphi : \Gamma \rightarrow \Gamma'$ an isomorphism. By Theorem 4.4 and Corollary 4.15 we may assume that Γ' is circulant and, as \mathbb{Z}_{pq} is a CI-group with respect to graphs, there exists $\delta \in \text{Aut}(\Gamma)$ such that $\delta^{-1} \varphi^{-1} \langle \rho, \tau_1 \rangle \varphi \delta = \langle \rho, \tau_1 \rangle$, and $\varphi \delta(v_j^i) = v_{s^{-1}j}^{r^{-1}i}$ (as $\varphi \delta \in \text{Aut}(\mathbb{Z}_{pq})$). Let $\varphi_1 = \varphi \delta$. Then $\varphi_1^{-1} \langle \rho \rangle \varphi_1 = \langle \rho \rangle$ and $\varphi_1^{-1} \tau \varphi_1(v_j^i) = v_{\alpha_j^i}^{i+r}$. If $q = 2$, then $\varphi_1^{-1} \tau \varphi_1 = \tau$ so that $\varphi_1^{-1} \langle \rho, \tau \rangle \varphi_1 = \langle \rho, \tau \rangle$. If $q = 3$, then $r = 1$ or $r = 2$. If $r = 1$, then $\varphi_1^{-1} \langle \rho, \tau \rangle \varphi_1 = \langle \rho, \tau \rangle$. If $r = 2$, define $\iota : V \rightarrow V$ by $\iota(v_j^i) = v_{-j}^{-i}$. As Γ is circulant, $\iota \in \text{Aut}(\Gamma)$. Then $\iota^{-1} \varphi_1^{-1} \tau \varphi_1 \iota = \tau$, so that $\iota^{-1} \varphi_1^{-1} \langle \rho, \tau_1 \rangle \varphi_1 \iota = \langle \rho, \tau \rangle$. \square

4.5 ISOMORPHISM CLASSES OF METACIRCULANT GRAPHS

We now determine an algorithm to explicitly calculate the isomorphism classes of (q, p) -metacirculant graphs. First we prove a lemma in more generality than is necessary for our purposes, characterizing the isomorphism class of a vertex-transitive combinatorial object in some circumstances.

Let X be a vertex-transitive combinatorial object of order m , G a transitive subgroup of $\text{Aut}(X)$, $C_G = \{\phi^{-1} G \phi : \phi \in S_n\}$, and $X_G = \{\phi^{-1} G \phi : \phi^{-1} G \phi \leq$

$\text{Aut}(X)\}$. Let S_n act on C_G by conjugation and denote the permutation group induced by this action as Ω . Let $\alpha_0 = 1$, the identity permutation in S_n , and $\alpha_1, \dots, \alpha_m \in S_n$ such that $G \leq \text{Aut}(\alpha_i(X))$, $0 \leq i \leq m$, $\alpha_j(X) \neq X$ for any $j \neq 0$, $\alpha_i(X) \neq \alpha_j(X)$ for any $i \neq j$, and if $\alpha \in S_n$ such that $G \leq \alpha(X)$ then $\alpha(X) = \alpha_i(X)$ for some $0 \leq i \leq m$.

Assume that X_G is a (possibly trivial) block of Ω . Let $X_G^0 = X_G$, and denote by $X_G^0, X_G^1, \dots, X_G^r$ all blocks conjugate to X_G^0 in Ω . Let $\beta_i \in S_n$ such that $\beta_i^{-1} X_G^0 \beta_i = X_G^i$, $1 \leq i \leq r$ and $\beta_0 = 1$.

Lemma 4.17. If X_G is a (possibly trivial) block of Ω , then the isomorphism class of X is $\cup_{i=0}^r \cup_{j=0}^m \beta_i \alpha_j(X)$, and if $a \neq b$ or $c \neq d$, then $\beta_a \alpha_c(X) \neq \beta_b \alpha_d(X)$.

Proof. Fix β_i , $0 \leq i \leq r$ and α_j , $0 \leq j \leq m$ as above. Clearly $\beta_i \alpha_j(X) \cong X$ for all $0 \leq i \leq r$, $0 \leq j \leq m$. Conversely, let Y be a combinatorial object isomorphic to X , with $\beta : X \rightarrow Y$ an isomorphism. If $G \leq \text{Aut}(Y)$, then $G \leq \text{Aut}(\beta_0^{-1}(Y))$ and there exists $0 \leq j \leq m$ such that $\alpha_j^{-1} \beta_0^{-1}(Y) = X$. Thus $Y = \beta_0 \alpha_j(X)$. If $G \not\leq \text{Aut}(Y)$, then $\beta^{-1}(X_G^0) \beta = X_G^i$ for some $1 \leq i \leq r$. Then $G \leq \beta_i^{-1}(Y)$ and so there exists $0 \leq j \leq m$ such that $\alpha_j^{-1} \beta_i^{-1}(Y) = X$. Thus $\beta_i \alpha_j(X) = Y$ as required. Finally, the last statement follows immediately from the definitions of β_i, α_j . \square

Clearly, by Theorem 3.7 we have

Corollary 4.18. Let Γ be a metacirculant graph of order pq such that Γ is not isomorphic to a circulant graph and $\text{Aut}(\Gamma)$ admits a complete block system of q blocks of size p . Then Γ_G is a block of Ω .

Theorem 4.19. If X is an n -CI-object for G and $\text{Aut}(G)_* \leq N_{S_V}(\text{Aut}(X))$, then X_G is a block of Ω .

Proof. Let $\delta \in S_V$ such that there exists a transitive subgroup G' of $\text{Aut}(X)$ isomorphic to G and $G' \leq \delta^{-1}\text{Aut}(X)\delta$. We will show that $\delta \in N_{S_G}(\text{Aut}(X))$. As X is an n -CI-object for G , there exists $\phi \in \text{Aut}(X)$ such that $\phi^{-1}G'\phi = G$. Now $G' \leq \text{Aut}(\delta(X))$, so $G \leq \text{Aut}(\phi\delta(X))$. Hence there exists $\alpha \in \text{Aut}(G)_*$ such that $\alpha_*(X) = \phi\delta(X)$, and as $\text{Aut}(G)_* \leq N_{S_{G_-}}(\text{Aut}(X))$, $\text{Aut}(\alpha_*(X)) = \text{Aut}(\phi\delta(X))$. As $\phi \in \text{Aut}(X)$, $\phi^{-1}\text{Aut}(X)\phi = \text{Aut}(X)$, and $\phi^{-1}\text{Aut}(X)\phi = \text{Aut}(\delta(X))$. Hence $\text{Aut}(\delta(X)) = \text{Aut}(X)$ and so $\delta^{-1}\text{Aut}(X)\delta = \text{Aut}(X)$. Thus $\delta \in N_{S_{G_-}}(\text{Aut}(x))$.

□

Corollary 4.20. Let Γ be a circulant graph of order pq , and $G = \mathbb{Z}_{pq}$. Then Γ_G is a block of Ω .

Proof. By [2], Γ is a CI-object for \mathbb{Z}_{pq} . Thus by Theorem 4.19 it suffices to show that $\text{Aut}(\mathbb{Z}_{pq}) \leq N_{S_G}(\text{Aut}(\Gamma))$. By Theorem 3.6, $\text{Aut}(\Gamma) = S_{pq}$, $S_p \times S_q$, $S_p \times A$, $S_q \times B$, $\text{Aut}(\Gamma) \leq A \times B$, $A \leq N(q)$, $B \leq N(p)$, or $\text{Aut}(\Gamma) = \text{Aut}(\Gamma_1) \wr \text{Aut}(\Gamma_2)$, where Γ_1 and Γ_2 are circulant graphs of order p and q respectively. Note that $\text{Aut}(G) \leq N(p) \times N(q)$, and that $N_{S_{pq}}(S_{pq}) = S_{pq}$, $S_p \times S_q \leq N_{S_{pq}}(S_p \times S_q)$,

$S_p \times GF(q) \leq N_{S_{pq}}(S_p \times A)$, $S_q \times N(p) \leq N_{S_{pq}}(S_q \times B)$, and $N(p) \times N(q) \leq N_{S_{pq}}(A \times B)$. Hence the result follows in the preceding cases.

If Γ' is a circulant graph of order r , a prime, then $\text{Aut}(\Gamma') = S_r$ or $\text{Aut}(\Gamma') \leq N(r)$. Hence if $\text{Aut}(\Gamma) = \text{Aut}(\Gamma_1) \wr \text{Aut}(\Gamma_2)$, $\text{Aut}(\Gamma) = S_p \wr S_q$, $S_p \wr A$, $B \wr S_q$, or $B \wr A$ where A and B are as above. In all of the these cases, we have that $N(p) \wr N(q) \leq N_{S_{pq}}(\text{Aut}(\Gamma))$. Then, as $N(p) \times N(q) \leq N(p) \wr N(q)$, $\text{Aut}(G) \leq N_{S_{pq}}(\text{Aut}(\Gamma))$, and the corollary follows. \square

CHAPTER 5

CIRCULANT GRAPHS OF ORDER A PRIME POWER

We now consider circulant graphs of order p^k . As mentioned in the introduction, the automorphism groups of circulant graphs of order p^k have already been determined, and the isomorphism problem solved by for $p \geq 3$ by Klin and Pöschel [36] and for $p = 2$, by Klin, Najmark, and Pöschel [38]. Let G be a transitive permutation group on \mathbb{Z}_n that contains a regular copy of \mathbb{Z}_n (an n -cycle). Schur [58] observed that from each such G , one can generate a subring $\mathcal{R}(G, \mathbb{Z}_n)$ of the group ring $\mathcal{R}(\mathbb{Z}_n)$, and Klin and Pöschel proved that given a subring S of $\mathcal{R}(\mathbb{Z}_n)$, there exists $G \leq S_n$ such that G contains a regular copy of \mathbb{Z}_n and $\mathcal{R}(G, \mathbb{Z}_n) = S$. Furthermore, $\mathcal{R}(G, \mathbb{Z}_n) = \mathcal{R}(\text{cl}(G), \mathbb{Z}_n)$. Hence by classifying all subrings of the group ring of \mathbb{Z}_n , one essentially classifies all 2-closed groups of S_n that contain a regular copy of \mathbb{Z}_n , and hence all automorphism groups of circulant graphs of order n . Our approach is quite different from that used by Klin and Pöschel, and will also give an extension of Burnside's Theorem (Theorem 2.4) to a significantly larger class of p -groups. It seems likely that the reason Klin and Pöschel did not also extend Burnside's Theorem is that their technique does not differentiate between a group G and its 2-closure.

Let $\tau_k : \mathbb{Z}_{p^k} \rightarrow \mathbb{Z}_{p^k}$ by $\tau_k(i) = i + 1$. Clearly τ_k admits a (possibly trivial) complete block system \mathcal{B}_i of p^i blocks of size p^{k-i} , where the blocks of \mathcal{B}_i are formed by the orbits of τ^{p^i} , $0 \leq i \leq k$. Let $\mathcal{B}_i = \{B_{i,j} : j \in \mathbb{Z}_{p^i}, \text{ and } j \in B_{i,j}\}$. Let $1 <$

$i \leq j < k$ and $P_{i,j}(\tau_k) = \langle \tau_k^{p^j} |_B : B \in \mathcal{B}_i \rangle$. Let $A \subset \mathbb{Z}_k^+ \times \mathbb{Z}_k^+$ such that if $(i, j) \in A$, then $i \leq j$. We say that A is a *2-closed index*, and if there exists $(i, j) \in A$ such that $i = j$, then A is a *wreath 2-closed index*. Let $\Pi_A(\tau_k) = \langle \tau_k, P_{i,j}(\tau_k) : (i, j) \in A \rangle$. Note that A is not necessarily unique. That is, there exists $A \neq A'$, both 2-closed indexes, such that $\Pi_A(\tau_k) = \Pi_{A'}(\tau_k)$ for $k \geq 3$. It is not difficult to see, however, that there is a unique minimal 2-closed index A such that if $\Pi_{A'}(\tau_k) = \Pi_A(\tau_k)$ then $|A'| \geq |A|$. We call such a 2-closed index *reduced*. Define $\pi_i : \Pi_A(\tau_k) \rightarrow S_{p^k}$ by $\pi_i(\gamma) = \gamma/\mathcal{B}_i$.

We will eventually show that if $p \geq 3$, that a Sylow p -subgroup of a 2-closed group is of the form $\Pi_A(\tau_k)$, for some 2-closed index, and the class of p -groups for which Burnside's Theorem can be extended is all $\Pi_A(\tau_k)$, with A a nonwreath 2-closed index.

5.1 THE EXTENSION OF BURNSIDE'S THEOREM

We first make some elementary observations about the groups $\Pi_A(\tau_k)$.

Lemma 5.1. Let $A \subset \mathbb{Z}_k^+ \times \mathbb{Z}_k^+$ be a 2-closed index and τ_k a p^k -cycle, $k \geq 2$. Then

- (i) $\Pi_A(\tau_k)/\mathcal{B}_i = \Pi_{A'}(\tau_k/\mathcal{B}_i)$, for some 2-closed index $A' \subset \mathbb{Z}_i^+ \times \mathbb{Z}_i^+$,
- (ii) $\text{Stab}_{\Pi_A(\tau_k)}(\mathcal{B}_{i,0}) = \text{Ker}(\pi_i)|_{\mathcal{B}_{i,0}} = \Pi_{A''}(\tau_k^{p^i}|_{\mathcal{B}_{i,0}})$ for some 2-closed index $A'' \subset \mathbb{Z}_{k-i}^+ \times \mathbb{Z}_{k-i}^+$,
- (iii) if A is a wreath 2-closed index, then $\Pi_A(\tau_k) = \Pi_{A'}(\tau_k^i/\mathcal{B}_i) \wr \Pi_{A''}(\tau_k^{p^i}|_{\mathcal{B}_{i,0}})$, where A' and A'' are 2-closed indexes, and

(iv) if A is nontrivial, then if $\gamma \in \Pi_A(\tau_k)$ such that $\pi_1(\gamma) \neq 1$, then $|\gamma| = p^k$.

Proof. (i) Observe that τ_k permutes the blocks of \mathcal{B}_i as a p^i -cycle. Then $\langle \tau_k / \mathcal{B}_i \rangle$ admits a complete block system \mathcal{B}'_j of p^j blocks of size p^{i-j} , where the blocks of \mathcal{B}'_j are formed by the orbits of $(\tau_k / \mathcal{B}_i)^{p^j}$. Note that if $m \geq i$, then $P_{\ell,m}(\tau_k) \leq \text{Ker}(\pi_i)$ and hence $\pi_i(P_{\ell,m}) = 1$. If $m < i$, then $P_{\ell,m}(\tau_k) / \mathcal{B}_i = P_{\ell,m-i}(\tau_k / \mathcal{B}_i)$. Let $A' = \{(\ell, m - i) : (\ell, m) \in A \text{ and } m < i\}$. Then $\Pi_A(\tau_k) / \mathcal{B}_i = \Pi_{A'}(\tau_k / \mathcal{B}_i)$.

(ii) By the definition of $\Pi_A(\tau_k)$, if $\gamma \in \text{Stab}_{\Pi_A(\tau_k)}(B_{i,0})$, then there exists $\gamma' \in \text{Ker}(\pi_i)$ such that $\gamma|_{B_{i,0}} = \gamma'|_{B_{i,0}}$. Hence it suffices to show that

$$\text{Ker}(\pi_i)|_{B_{i,0}} = \Pi_{A''}(\tau_k^{p^i}|_{B_{i,0}})$$

for some 2-closed $A'' \subset \mathbb{Z}_{k-i}^+ \times \mathbb{Z}_{k-i}^+$. Note that $\langle \tau_k^{p^i} |_{B_{i,0}} \rangle$ admits a complete block system \mathcal{B}'_j of p^j blocks of size p^{k-i-j} , $1 \leq j \leq k-i-1$, where the blocks of \mathcal{B}'_j are formed by the orbits of $(\tau_k^{p^i} |_{B_{i,0}})^{p^j}$. Observe that $\text{Ker}(\pi_i) = \langle \tau^{p^i}, P_{\ell,m}(\tau_k) : (\ell, m) \in A \text{ and } m \geq i \rangle$. If $m \geq i$ and $\ell \leq i$, then $P_{\ell,m}(\tau_k)|_{B_{i,0}} = \langle \tau_k^{p^m} \rangle|_{B_{i,0}}$. If $m \geq i$ and $\ell > i$, then

$$P_{\ell,m}(\tau_k)|_{B_{i,0}} = \langle (\tau_k^{p^i}|_{B_{i,0}})^{p^{m-i}}|_B : B \in \mathcal{B}'_{\ell-i} \rangle.$$

Hence if $A'' = \{(\ell - i, m - i) : (\ell, m) \in A \text{ and } m \geq i\}$, then $\text{Ker}(\pi_i)|_{B_{i,0}} = \Pi_{A''}(\tau^{p^i})|_{B_{i,0}}$.

(iii) If A is a wreath 2-closed index, then let $1 \leq i \leq k-1$ such that $(i, i) \in A$. By (i) and (ii), $\Pi_A(\tau_k) / \mathcal{B}_i = \Pi_{A'}(\tau_k / \mathcal{B}_i)$ and $\text{Ker}(\pi_1)|_{B_{i,0}} = \Pi_{A''}(\tau_k^{p^i}|_{B_{i,0}})$. As $(i, i) \in A$, $\text{Ker}(\pi_i) = 1_{S_{p^i}} \wr \Pi_{A''}(\tau_k^{p^i}|_{B_{i,0}})$, and the result follows.

(iv) We proceed by induction on k . If $k = 2$, $\Pi_A(\tau_2) = \langle \tau_2 \rangle$ and the result is trivial. Assume $k \geq 2$ and if A is not a wreath 2-closed index then if $\gamma \in \Pi_A(\tau_k)$ such that $\pi_1(\gamma) \neq 1$, the $|\gamma| = p^k$. Let $A \subset \mathbb{Z}_{k+1}^+ \times \mathbb{Z}_{k+1}^+$ be a nonwreath 2-closed index, and $\gamma \in \Pi_A(\tau_{k+1})$ such that $\pi_1(\gamma) \neq 1$. As A is a nonwreath 2-closed index, $\gamma^p \in \text{Ker}(\pi_1)$ and $\gamma^p/B_2 \neq 1$. Then result then follows by (ii) and the induction hypothesis. \square

We now prove two technical lemmas needed to prove the extension of Burnside's Theorem.

Lemma 5.2. Let Π a Sylow p -subgroup of S_{p^k} , $k \geq 2$ that contains τ_k . Then Π admits a complete block system \mathcal{B} of p^{k-1} blocks of size p , where the blocks of \mathcal{B} are formed by the orbits of $\tau_k^{p^{k-1}}$. Define $\pi : \Pi \rightarrow S_{p^{k-1}}$ by $\pi(\gamma) = \gamma/\mathcal{B}$. If $|\Pi| > p^{k+1}$, then $|\text{Ker}(\pi_1)| > p$ or $\Pi \leq N(p^k)$ and $p = 2$.

Proof. We proceed by induction on k . If $k = 2$, the result is trivial, so assume $k \geq 3$ and that the result holds for every Sylow p -subgroup of $S_{p^{k-1}}$ that contains τ_k such that $|\Pi| \geq p^k$. Let $\tau = \tau_k$ and Π be a Sylow p -subgroup of S_{p^k} that contains τ . As Π is a Sylow p -subgroup, there exists $\alpha \in \text{C}(\Pi)$, the center of Π , such that α is semiregular of order p . Then the orbits of α form a complete block system \mathcal{B} of p^{k-1} blocks of size p . As $\tau \in \Pi$, we may assume $\alpha \in \langle \tau \rangle$, and hence that $\alpha \in \langle \tau^{p^{k-1}} \rangle$. Let π be as above and assume $|\Pi| > p^{k+1}$ but $|\text{Ker}(\pi)| = p$. Then $\text{Ker}(\pi) = \langle \tau^{p^{k-1}} \rangle$ and $\text{Im}(\pi)$ is a Sylow p -subgroup of $S_{p^{k-1}}$ that contains τ/\mathcal{B} , and

$|\text{Im}(\pi)| \geq p^k$. Hence by the induction hypothesis, $\text{Im}(\pi)$ admits a complete block system \mathcal{C} of p^{k-2} blocks of size p , and if $\pi' : \text{Im}(\pi) \rightarrow S_{p^{k-2}}$ by $\pi'(\gamma) = \gamma/\mathcal{C}$, then either $|\text{Ker}(\pi')| \geq p^2$ or $\Pi \leq N(p^{k-1})$ and $p = 2$.

If $\Pi \leq N(p^{k-1})$ and $p = 2$, then $\langle \tau \rangle \triangleleft \Pi$ and $p = 2$. If $|\text{Ker}(\pi')| \geq p^2$, then first observe that the blocks of \mathcal{C} are formed by the orbits of $\pi(\tau^{p^{k-2}})$. Hence Π admits a complete block system \mathcal{D} of p^{k-2} blocks of size p^2 , where the blocks of \mathcal{D} are formed by the orbits of $\tau^{p^{k-2}}$. Define $\pi^* : \Pi \rightarrow S_{p^{k-2}}$ by $\pi^*(\gamma) = \gamma/\mathcal{D}$. As $|\text{Ker}(\pi')| \geq p^2$, $|\text{Ker}(\pi^*)| \geq p^3$. Hence there exists $\gamma \in \text{Ker}(\pi^*)$ such that $\gamma \notin \langle \tau^{p^{k-2}} \rangle$. Note that each element of \mathbb{Z}_{p^k} may be written uniquely in the form $i + jp^{k-2}$, where $i \in \mathbb{Z}_{p^{k-2}}$ and $j \in \mathbb{Z}_{p^2}$. Denote the blocks of \mathcal{D} by $D_0, D_1, \dots, D_{p^{k-2}-1}$, where $i + 0 \cdot p^{k-2} \in D_i$. Now, $\text{Ker}(\pi^*) \leq 1_{S_{p^{k-2}}} \wr (C_p \wr C_p)$, where C_p is a cyclic group of order p , so that $\pi(\text{Ker}(\pi^*)) \leq 1_{S_{p^{k-2}}} \wr C_p$ and hence $\pi(\text{Ker}(\pi^*))$ is abelian. Hence $\gamma^{-1}\tau^{p^{k-2}}\gamma/\mathcal{B} = \tau^{p^{k-2}}/\mathcal{B}$. Further, $\gamma^{-1}\tau^{p^{k-2}}\gamma\tau^{-p^{k-2}} \in \text{Ker}(\pi)$ so that $\gamma^{-1}\tau^p\gamma \in \langle \tau^{p^{k-2}} \rangle$ and $\gamma \in N_{S_{p^k}}(\langle \tau^{p^{k-2}} \rangle)$. Hence $\gamma|_{D_i} \in N_{S_{D_i}}(\langle \tau^{p^{k-2}} \rangle|_{D_i})$ so that $\gamma(i + jp^{k-2}) = i + (\beta j + b_i)p^{k-2}$, $\beta \in \mathbb{Z}_{p^2}^*$, $b_i \in \mathbb{Z}_{p^2}$. As $|\gamma| = p$ or p^2 , $|\beta| = 1$ or p . If $|\beta| = 1$, then as $\gamma/\mathcal{B} \neq 1$ and $\gamma/\mathcal{B} \notin \langle \tau^{p^{k-2}} \rangle$, $|\text{Ker}(\pi)| \geq p^2$. If $|\beta| = p$, then $\beta = (p+1)^a$ for some $a \in \mathbb{Z}_p^*$. Now, $\gamma^{-1}\tau\gamma \in \langle \tau \rangle$ or $\gamma^{-1}\tau\gamma \notin \langle \tau \rangle$. If $\gamma^{-1}\tau\gamma \notin \langle \tau \rangle$, then $\gamma' = \gamma^{-1}\tau\gamma\tau^{-1} \in \text{Ker}(\pi')$ and $\gamma'(i + jp^{k-2}) = i + (j + c_i)p^{k-2}$, $c_i \in \mathbb{Z}_{p^2}$, and as $\tau^{p^{k-2}} \in \text{Ker}(\pi^*)$, we may assume that $c_0 = 0$. As $\gamma^{-1}\tau\gamma \notin \langle \tau \rangle$, some $c_i \neq 0$ so that either $\gamma' \in \text{Ker}(\pi)$, or $(\gamma')^p \neq 1$ and $(\gamma')^p \in \text{Ker}(\pi)$. Hence $|\text{Ker}(\pi)| \geq p^2$. If $\gamma \in N_{S_{p^k}}(\langle \tau \rangle)$, then if $p \neq 2$, $\gamma^p \in \text{Ker}(\pi)$, $\gamma^p \notin \langle \tau^{p^{k-1}} \rangle$, so that $|\text{Ker}(\pi)| \geq p^2$.

□

Let $\beta \in \mathbb{Z}_{p^k-\ell}^*$ such that if $\gamma : \mathbb{Z}_{p^k-\ell} \rightarrow \mathbb{Z}_{p^k-\ell}$ by $\gamma(i) = \beta i$, then $\gamma\tau_{k-\ell}\gamma^{-1} = \tau_{k-\ell}^{1+p^m}$. Regard \mathbb{Z}_{p^k} as $\{i + jp^\ell : i \in \mathbb{Z}_{p^\ell}, j \in \mathbb{Z}_{p^{k-\ell}}\}$, and define $\gamma_{\ell,m} : \mathbb{Z}_{p^k} \rightarrow \mathbb{Z}_{p^k}$ by

$$\gamma_{\ell,m}(i + jp^\ell) = i + \beta jp^\ell.$$

Lemma 5.3. Let $k \geq 2$ and A a reduced nonwreath 2-closed index. Let $p^t = |\Pi_A(\tau_k)|$. Then $N_{S_{p^k}}(\Pi_A(\tau_k)) = \langle \gamma_{\ell,m}, N(p^k), \Pi_A(\tau_k) : (\ell, m) \in A \rangle$. Furthermore, $|N_{S_{p^k}}(\Pi_A(\tau_k))| = (p-1)p^{k-1}p^t$.

Proof. We first show that $|N_{S_{p^k}}(\Pi_A(\tau_k))| = (p-1)p^{k-1}p^t$.

There are $p^k!/[(p-1)p^{k-1}p^k]$ subgroups of S_{p^k} conjugate to $\langle \tau_k \rangle$ in S_{p^k} . Clearly $|\Pi_A(\tau_k)| = p^t$ for some $t \geq k$, and if $\tau' \in \Pi_A(\tau_k)$ such that $\tau'/\mathcal{B}_1 \neq 1$, then by Lemma 5.1, $|\tau'| = p^k$. Further, we may assume that $\tau'/\mathcal{B}_1 = \tau_k/\mathcal{B}_1$ and hence that $\tau'\tau_k^{-1} \in \text{Ker}(\pi_1)$. Now, $|\text{Ker}(\pi_1)| = p^{t-1}$ so that there are p^{t-1} distinct p^k cycles τ^* in $\Pi_A(\tau_k)$ such that $\tau^*/\mathcal{B}_1 = \tau_k/\mathcal{B}_1$. Note that $\langle \tau' \rangle$ contains p^{k-1} p^k -cycles τ^* such that $\tau^*/\mathcal{B}_1 = \tau_k/\mathcal{B}_1$. We conclude that there are p^{t-k} subgroups contained in $\Pi_A(\tau_k)$ that are conjugate to $\langle \tau_k \rangle$ in S_{p^k} .

Let S_{p^k} act by conjugation on $\Delta = \{H \leq S_{p^k} : H \text{ is conjugate to } \langle \tau_k \rangle \text{ in } S_{p^k}\}$. Denote the resulting permutation group by P . Let $X = \{H \in \Delta : H \leq \Pi_A(\tau_k)\}$. Let $\alpha \in S_{p^k}$ such that $\alpha X \alpha^{-1} \cap X \neq \emptyset$. Then there exists $\langle \tau' \rangle \in X$ such that $\alpha \langle \tau' \rangle \alpha^{-1} \in X$. Note that $\tau'^{p^{k-1}} \in \langle \tau_k^{p^{k-1}} \rangle$ so that $\alpha \tau'^{p^{k-1}} \alpha^{-1} \in \langle \tau_k^{p^{k-1}} \rangle$. We conclude that $\text{Ker}(\pi_{k-1}) \leq \alpha \Pi_A(\tau_k) \alpha^{-1}$. Note that $\tau'^{p^{k-2}} \in \langle \tau_k^{p^{k-2}}, \text{Ker}(\pi_{k-1}) \rangle$ and hence $\text{Ker}(\pi_{k-2}) \leq \alpha \Pi_A(\tau_k) \alpha^{-1} = \Pi_A(\tau_k)$. Arguing inductively, we have that

$\alpha \Pi_A(\tau_k) \alpha^{-1} = \Pi_A(\tau_k)$. Thus X is a block of P , and so the number of subgroups conjugate to $\Pi_A(\tau_k)$ in S_{p^k} is the number of blocks conjugate to X in P . Hence there are

$$\frac{p^k!}{(p-1)p^{k-1}p^t} = [S_{p^k} : N_{S_{p^k}}(\Pi_A(\tau_k))]$$

subgroups of S_{p^k} conjugate to $\Pi_A(\tau_k)$ in S_{p^k} and $|N_{S_{p^k}}(\Pi_A(\tau_k))| = (p-1)p^{k-1}p^t$.

Let $(\ell, m) \in A$, and $\alpha \in \Pi_A(\tau_k)$. If $\alpha \in \text{Ker}(\pi_\ell)$, then clearly $\gamma_{\ell,m}^{-1} \alpha \gamma_{\ell,m} \in \text{Ker}(\pi_\ell)$. If $\alpha \notin \text{Ker}(\pi_\ell)$, then by Lemma 5.1, $\text{Im}(\pi_\ell) = \Pi_{A'}(\tau_\ell)$ for some nonwreath 2-closed index A' . Assume for the moment that $\gamma_{\ell,m}^{-1} \tau_k \gamma_{\ell,m} \in \Pi_A(\tau_k)$. Note that $\alpha = \tau_k^a \Pi_{\alpha_{i,j}} \gamma$, where $a \in \mathbb{Z}_{p^k}$, $\alpha_{i,j} \in P_{i,j}$, $(i, j) \in A$ with $i < \ell$, and $\gamma \in \text{Ker}(\pi_\ell)$. Now, $P_{i,j} = \langle \tau_k^{p^j} |_B : B \in \mathcal{B} \rangle$, and $\gamma_{\ell,m}^{-1} \tau_k^{p^j} \gamma_{\ell,m} \in \Pi_A(\tau_k)$. Also $\gamma_{\ell,m}^{-1} \tau_k^{p^j} \gamma_{\ell,m} \in \text{Ker}(\pi_j)$ and $\text{Ker}(\pi_j) = \langle P_{r,s} : (r, s) \in A \text{ and } k \geq j \rangle$. We conclude that $\gamma_{\ell,m}^{-1} \tau_k^{p^j} \gamma_{\ell,m} \in \Pi_A(\tau_k)$ for every $B \in \mathcal{B}$. Hence it suffices to show that $\gamma_{\ell,m}^{-1} \tau_k \gamma_{\ell,m} \in \Pi_A(\tau_k)$.

We consider \mathbb{Z}_{p^k} as $\{i + jp^\ell : i \in \mathbb{Z}_{p^{k-\ell}}\}$. Hence $\tau_k(i + jp^\ell) = i + 1 + \sigma_k(j)p^\ell$, where $\sigma_i(j) = j$ if $i \neq p^\ell - 1$ and $\sigma_{p^\ell-1}(j) = j + 1$. Then $\gamma_{\ell,m}^{-1} \tau_k \gamma_{\ell,m}(i + jp^\ell) = i + 1 + jp^\ell$ if $i \neq p^\ell - 1$ and $\gamma_{\ell,m}^{-1} \tau_k \gamma_{\ell,m}(p^\ell - 1 + jp^\ell) = p^\ell + (j + 1 + p^m)p^\ell$. As $(\ell, m) \in A$, $\gamma_{\ell,m}^{-1} \tau_k \gamma_{\ell,m} \in \Pi_A(\tau_k)$ and thus $\gamma_{\ell,m} \in N_{S_{p^k}}(\Pi_A(\tau_k))$.

Let $G = \langle \gamma_{\ell,m}, \Pi_A(\tau_k), N(p^k) : (\ell, m) \in A \rangle$. We will show that $|G| = (p-1)p^{k-1}p^t$ and hence that $G = N_{S_{p^k}}(\Pi_A(\tau_k))$. Recall that $|N(p^k)| = (p-1)p^{k-1}p^k$ and that $|\Pi_A(\tau_k)| = p^t$. Let $(\ell, m) \in A$ such that ℓ is minimal. Let $\delta \in N(p^k)$ such that $\delta(i) = (1+p)i$. Then $|\delta| = p^{k-1}$ and $\delta p^{m-1} \in \Pi_A(\tau_k)$. Hence

$$\frac{|N_{S_p^k}(\Pi_A(\tau_k))|}{|\langle \Pi_A(\tau_k), N(p^k) \rangle|} = \frac{(p-1)p^{k-1}p^t}{(p-1)p^t p^{k-m}} = p^m.$$

Let $(i, j) \in A$ and $(k, \ell) \in A$ such that there exists no $(r, s) \in A$ with $i < r < k$.

Then $\gamma_{i,j}^{p^{j-i}} \in P_{k,\ell}(\tau_k)$. Hence

$$\frac{|\langle \gamma_{\ell,m} : (\ell, m) \in A \rangle|}{|\langle \gamma_{\ell,m} : (\ell, m) \in A \rangle \cap \Pi_A(\tau_k)|} = p^m$$

and so $|G| = (p-1)p^{k-1}p^t$ as required. \square

Lemma 5.4. Let G be a transitive group of degree n acting on \mathbb{Z}_n that contains an n -cycle τ . Assume that G admits a complete block system \mathcal{B} of m blocks of size p , $mp = n$. Define $\pi : G \rightarrow S_m$ by $\pi(\gamma) = \gamma/\mathcal{B}$. If $\pi(G)$ is doubly transitive, \mathbb{Z}_m is pronormal in $\pi(G)$, $p|m$, and $m \neq 2, 3$, then $\text{Ker}(\pi) \not\cong \mathbb{Z}_p$.

Proof. Assume without loss of generality that $\tau(i) = i + 1$. Then $\tau^m \in \text{Ker}(\pi)$. Suppose $\text{Ker}(\pi) \cong \mathbb{Z}_p$. Then $\text{Ker}(\pi) = \langle \tau^m \rangle$. As $\pi(G)$ is doubly transitive, there exists $\beta \in G$ such that $|\pi(\beta)| = m$ but $\pi(\beta) \notin \langle \pi(\tau) \rangle$. As \mathbb{Z}_m is pronormal in G there exists $\delta \in G$ such that $\pi(\delta^{-1}\langle \beta \delta \rangle) = \pi(\langle \beta \rangle)$. Hence there exists $t \in \mathbb{Z}_m^*$ such that $\delta^{-1}\beta\delta\tau^t \in \text{Ker}(\pi)$. Let $\delta^{-1}\beta\delta\tau^t = \tau^{mr}$. Then $\delta^{-1}\beta\delta\tau^{t-mr} = 1$ and $t - mr \in \mathbb{Z}_m^*$. Hence $|\delta^{-1}\beta\delta| = n$ so that $|\beta| = n$.

As $\text{Ker}(\pi) = \langle \tau^m \rangle$, $\beta^m \in \langle \tau^m \rangle$. Note that we may consider each element of \mathbb{Z}_n as being uniquely of the form $i + jm$, $i \in \mathbb{Z}_m$, $j \in \mathbb{Z}_p$, and that $\tau(i + jm) = i + 1 + \sigma_i(j)m$, where $\sigma_i : \mathbb{Z}_p \rightarrow \mathbb{Z}_p$, $\sigma_i(j) = j$ if $i \neq m-1$ and $\sigma_{m-1}(j) = j + 1$.

As $\langle \tau^m \rangle \triangleleft G$, $\beta(i + jm) = \delta(i) + \omega_i(j)m$, where $\delta \in S_m$, $\omega_i \in N(p)$. Hence $\omega_i(j) = \alpha_i j + b_i$, $\alpha_i \in \mathbb{Z}_p^*$, $b_i \in \mathbb{Z}_p$. If $\alpha_i \neq \alpha_j$ for any $i, j \in \mathbb{Z}_m$, then $p^2 \mid |\text{Ker}(\pi)|$. Hence $\omega_i(j) = \alpha j + b_i$. As $|\beta| = n$, either $\alpha = 1$ or $(|\alpha|, p) = p$. As $|\mathbb{Z}_p^*| = p - 1$, $\alpha = 1$. As $|\beta| = n$ and $\beta^m \in \langle \tau^m \rangle$, $\beta^m = \tau^{mc}$, $(c, p) = 1$. Hence $\sum_{i=0}^{m-1} b_i \equiv c \pmod{p}$. Let $a \in \mathbb{Z}_m$ such that $\tau^a \beta / \mathcal{B}$ has a fixed point. Denote by $\mathcal{O}_1, \mathcal{O}_2, \dots, \mathcal{O}_r$ the orbits of $\tau^a \beta / \mathcal{B}$, and let $d_i = |\mathcal{O}_i|$. Let $\tau^a \beta(i + jm) = \delta'(i) + (j + c_i)m$, $\delta' \in S_m$, $c_i \in \mathbb{Z}_p$. Then $\sum_{i=0}^{m-1} c_i = \sum_{i=0}^{m-1} b_i + a$. Now, $\tau^a \beta / \mathcal{B}$ has a fixed block B' and $\tau^a \beta|_{B'} = \tau^{my}$, $y \in \mathbb{Z}_p$. Further, if $B_d \in \mathcal{O}_i$, then $(\tau^a \beta)^{d_i}|_{B_d} = \tau^{myd_i}$. Hence

$$\sum_{i=0}^{m-1} c_i \equiv \sum_{i=1}^r y d_i \equiv ym \pmod{p}$$

so that $\sum_{i=0}^{m-1} b_i \equiv -a \pmod{p}$. However, as $m \neq 2, 3$, there exists $d \neq a \pmod{p}$ such that $\tau^d \beta / \mathcal{B}$ has a fixed point. Analogous arguments then show that $\sum_{i=0}^{n-1} b_i \equiv d \pmod{p}$, a contradiction. \square

Note $N_{S_{p^k}}(\Pi_A(\tau_k))$ admits \mathcal{B}_1 as a complete block system. Further, if $\gamma \in N_{S_{p^k}}(\Pi_A(\tau_k))$ such that $|\gamma| \neq p^i$ for any i , then $\gamma / \mathcal{B}_1 \neq 1$.

Theorem 5.5. Let G be a transitive group of degree p^k and Π be a Sylow p -subgroup of G . If $\Pi = \Pi_A(\tau_k)$ for some nonwreath 2-closed index A , then G is doubly transitive or $\Pi \triangleleft G$.

Proof. We proceed by induction on k . If $k = 1$, the result follows from Burnside's Theorem. Assume $k \geq 2$ and that the result is true for all $1 \leq i \leq k - 1$. Let $\tau = \tau_k$

and G be as above with Sylow p -subgroup $\Pi = \Pi_A(\tau)$. As $\tau \in G$ and \mathbb{Z}_{p^k} is a Burnside group, G is doubly transitive or imprimitive. If G is doubly transitive, the result is trivial, so we assume that G admits a complete block system \mathcal{B}_i of p^i blocks of size p^{k-i} , where \mathcal{B}_i is defined as above. Define π_i as above and let $K = \text{Ker}(\pi_i)$. By Lemma 5.1, as $\Pi_A(\tau)$ is a Sylow p -subgroup of G , $\Pi_{A'}(\tau^{p^i}|_{B_{i,0}})$ is a Sylow p -subgroup of $\text{Ker}(\pi_i)|_{B_{i,0}}$, where A' is a nonwreath 2-closed index. Hence by the induction hypothesis, $\text{Ker}(\pi_i)|_{B_{i,0}}$ is doubly transitive or $\Pi_{A'}(\tau^{p^i}|_{B_{i,0}}) \triangleleft \text{Ker}(\pi_i)|_{B_{i,0}}$.

If $\text{Ker}(\pi_i)|_{B_{i,0}}$ is doubly transitive, define $\pi' : \text{Ker}(\pi_i) \rightarrow S_{p^{k-i}}$ by $\pi'(\gamma) = \gamma|_{B_{i,0}}$. If $\text{Ker}(\pi') \neq 1$, then there exists j such that $\text{Ker}(\pi')|_{B_{i,j}} \neq 1$. As $\text{Ker}(\pi_i)|_{B_{i,0}}$ is doubly transitive, $\text{Ker}(\pi')|_{B_{i,j}}$ is transitive, and hence contains a transitive subgroup of degree p^{k-i} and order p^{k-i+r} , $r \geq 0$. As $\Pi_{A'}(\tau^{p^i}|_{B_{i,j}})$ is a Sylow p -subgroup of K , $\text{Ker}(\pi')|_{B_{i,j}}$ contains a p^{k-i} cycle. Define an equivalence relation \equiv on \mathcal{B}_i by $B_{i,s} \equiv B_{i,t}$ if and only if whenever $\gamma \in K$, then $\gamma|_{B_{i,s}}$ is a p^{k-i} cycle if and only if $\gamma|_{B_{i,t}}$ is also a p^{k-i} cycle. As $\Pi_A(\tau)$ is a Sylow p -subgroup of G , the union of the equivalence classes of \equiv form a complete block system \mathcal{B}_ℓ of $\langle \tau \rangle$, $\ell \leq i$. Hence $(\ell, i) \in A$, and if $(m, i) \in A$, then $1 \leq m \leq \ell$. As A is a nonwreath 2-closed index, $\ell < i$. Let $J = \text{Stab}_G(\mathcal{B}_{\ell,0})|_{B_{\ell,0}}$. Then a Sylow p -subgroup of J is of the form $\Pi_{A''}(\tau^{p^\ell}|_{B_{\ell,0}})$ for some nonwreath 2-closed index A'' . As $K|_{B_{\ell,0}} \leq J$, it follows from the induction hypothesis that J is doubly transitive. However, as G admits a complete block system of p^i blocks of size p^{k-i} , J admits a complete block system of $p^{\ell-k+i}$ blocks of size p^{k-i} , a contradiction. Hence $\text{Ker}(\pi') = 1$.

If $\text{Ker}(\pi') = 1$, define an equivalence relation \equiv on \mathbb{Z}_{p^k} by $u \equiv v$ if and only if $\text{Stab}_K(u) = \text{Stab}_K(v)$. As $\text{Ker}(\pi') = 1$, $K \cong 1_{S_{p^i}} \times \text{Im}(\pi')$. As $\text{Im}(\pi')$ is doubly transitive, each equivalence class of \equiv contains at most one element of each $B_{i,j}$ and as $K \cong 1_{S_{p^i}} \times \text{Im}(\pi')$, each equivalence class of \equiv contains exactly one element of each $B_{i,j}$. However, each equivalence class of \equiv is a block of Π that is not formed by the orbits of τ to an appropriate power, a contradiction. Hence $H = \Pi_{A'}(\tau^{p^i}|_{B_{i,0}}) \triangleleft K|_{B_{i,0}}$.

If $H \triangleleft K|_{B_{i,0}}$, we first consider when $K|_{B_{i,0}} \neq H$. If $K|_{B_{i,0}} \neq H$, then by Lemma 5.3 $K|_{B_{i,0}} \leq N_{S_{p^{k-i}}}(H)$, $|N_{S_{p^{k-i}}}(H)| = (p-1)p^{k-1-i}p^t$, where $|H| = p^t$. As H is a Sylow p -subgroup of $K|_{B_{i,0}}$, there exists $\alpha \in K$ such that $\pi_{i+1}(\alpha) \neq 1$ and $(|\alpha|, p) = 1$. Then $\pi_{i+1}(G)$ has a Sylow p -subgroup $\Pi' = \Pi_{A''}(\tau/B_{i+1})$ that is not a wreath product. Clearly $\alpha \notin N_{S_{p^{k-i-1}}}(\Pi')$ and hence by the induction hypothesis, $\pi_{i+1}(G)$ is doubly transitive, contradicting the assumption that G admits a complete block system of p^i blocks of size p^{k-i} . Hence $K|_{B_{i,0}} = H$.

Now, if $\pi_i(\Pi_A(\tau)) \triangleleft \pi_i(G)$, then $\Pi_A(\tau) \triangleleft G$ and the result follows. If not, then by the induction hypothesis, $\pi_i(G)$ is doubly transitive. If $\pi_i(G)$ is doubly transitive, then G admits a (possibly trivial) complete block system \mathcal{B} of p^{i+1} blocks of size p^{k-i-1} , and so $\pi_{i+1}(G)$ admits a complete block system \mathcal{C} of p^i blocks of size p . Define $\pi : \pi_{i+1}(G) \rightarrow S_{p^i}$ by $\pi(\gamma) = \gamma/\mathcal{C}$. If $|\text{Ker}(\pi)| \neq p$, then by Lemma 5.2 $p^2 | |\text{Ker}(\pi)|$ and we conclude that $\Pi_A(\tau)$ is a nontrivial wreath product, a contradiction. If $|\text{Ker}(\pi)| = p$, then by Lemma 5.2 the Sylow p -subgroups of $\pi(\pi_{i+1}(G))$ are cyclic. If $k < i + 1$, then by the induction hypothesis, $\pi_{i+1}(G)$ is doubly transitive, a

contradiction. Hence $k = i + 1$, Π is cyclic, and $\pi = \pi_i = \pi_{k-1}$. It then follows from Lemma 5.4 that $\text{Ker}(\pi) \not\cong \mathbb{Z}_p$, a contradiction. \square

Corollary 5.6. Let $p = 2$ and A a nonwreath 2-closed index. Let G be a transitive group of degree 2^k with Sylow 2-subgroup $\Pi_A(\tau_k)$. Then G is doubly transitive or $G = \Pi_A(\tau_k)$.

Proof. By Theorem 5.5, $\Pi_A(\tau_k) \triangleleft G$ or G is doubly transitive. By Lemma 5.3, $|N_{S_{2^k}}(\Pi_A(\tau_k))| = 2^{k-1}2^t$, where $|\Pi_A(\tau_k)| = 2^t$. As $\Pi_A(\tau_k)$ is a Sylow 2-subgroup of G , the result follows. \square

We now will extend Burnside's Theorem to an even wider class of 2-groups. Define $\iota_k : \mathbb{Z}_{2^k} \rightarrow \mathbb{Z}_{2^k}$ by $\iota_k(i) = -i$.

Theorem 5.7. Let $\Pi = \langle \Pi_A(\tau_k), \iota_k \rangle$ for some nonwreath 2-closed index A . Let G be a transitive group of degree 2^k such that Π is a Sylow 2-subgroup of G . Then G is doubly transitive or $G = \Pi$.

Proof. We would first like to make some observations about Π . Note that Π admits $\mathcal{B}_0, \dots, \mathcal{B}_k$ as complete block systems. Define $\pi'_i : \Pi_A(\tau_k) \rightarrow S_{2^i}$ by $\pi'_i(\gamma) = \gamma/\mathcal{B}_i$. Note that if $i \geq 2$, then $\text{Ker}(\pi_i) = \text{Ker}(\pi'_i)$, and $\text{Ker}(\pi_1) = \langle \text{Ker}(\pi'_1), \iota \rangle$.

We proceed by induction on k . If $k = 2$, then $|\Pi| = 8$ and $|S_4| = 24$. Hence $G = \Pi$ or $G = S_4$ and the result follows. Assume $k \geq 2$ and the result holds for all $2 \leq i \leq k$. Let G be a transitive group of degree 2^{k+1} such that Π is a Sylow

2-subgroup of G , for some nonwreath 2-closed index A . As $\mathbb{Z}_{2^{k+1}}$ is a Burnside group, G is doubly transitive or imprimitive. If G is doubly transitive, then the result is trivial. If not, then G admits \mathcal{B}_i as a complete block system for some $1 \leq i \leq k$. Define $\pi : G \rightarrow S_{p^i}$ by $\pi(\gamma) = \gamma/\mathcal{B}_i$, and define $\pi' : \text{Ker}(\pi) \rightarrow S_{2^{k-i+1}}$ by $\pi'(\gamma) = \gamma|_B$, $B \in \mathcal{B}_i$, $0 \in B$. It follows by arguments in Theorem 5.5 that if $i \neq k$, then $\text{Im}(\pi')$ is not doubly transitive. If $i = k$, then $\text{Im}(\pi') \cong S_2 \cong \mathbb{Z}_2$, and hence that $\text{Ker}(\pi)$ is a 2-group. If $i \neq k$ and $i \neq 1$, then by comments above and Lemma 5.1, $\text{Im}(\pi') = \Pi_{A'}(\tau_{k-i+1})$ for some nonwreath 2-closed index A' . By Corollary 5.6, $\text{Ker}(\pi) = \text{Ker}(\pi'_i)$. It then follows by arguments in Theorem 5.5 and Lemma 5.3 that $\Pi = G$. We thus need only consider the cases where $i = k$ and $i = 1$.

If $i = 1$, then as $\text{Im}(\pi')$ is not doubly transitive, it is not difficult to check that $\pi'(\text{Ker}(\pi'_1)) \geq \langle \Pi_{A'}(\tau_k), \iota_k \rangle$ for some nonwreath 2-closed index A' . It then follows by the induction hypothesis and Lemma 5.3 that $\text{Ker}(\pi) = \text{Ker}(\pi'_1)$. As $|\text{Im}(\pi')| = 2$, $G = \Pi$.

If $i = k$, then $\pi(\Pi)$ is a Sylow 2-subgroup of $\text{Im}(\pi)$. By the induction hypothesis, $\pi(\Pi) = \text{Im}(\pi)$ or $\text{Im}(\pi)$ is doubly transitive. If $\pi(\Pi) = \text{Im}(\pi)$, then as $\text{Ker}(\pi) = \text{Ker}(\pi'_k)$, $\Pi = G$. If $\text{Im}(\pi)$ is doubly transitive, then either A is a nonwreath 2-closed index or $|\text{Ker}(\pi)| = 2$. If $|\text{Ker}(\pi)| = 2$, then as $|\Pi| \geq 2^{k+2}$, it follows by Lemma 5.2 that $\Pi \leq N(2^{k+1})$, and Π contains exactly one cyclic group of order 2^{k+1} . As $\Pi/\mathcal{B} = \langle \iota_k, \Pi_\emptyset(\tau_k) \rangle$, \mathbb{Z}_{2^k} is pronormal in $\text{Im}(\pi)$. Hence by Lemma 5.4, $\text{Ker}(\pi) \not\cong \mathbb{Z}_2$, a contradiction. \square

5.2 APPLICATIONS

We first prove a technical lemma that will be used to prove the main application of the extension of Burnside's theorem.

Lemma 5.8. Let G be a 2-closed transitive permutation group such that there exists $H \leq G$ such that H admits a complete block system of \mathcal{B} of m blocks of size p^k , p a prime. Assume that H is the maximal subgroup of G satisfying the above condition. Define $\pi : H \rightarrow S_m$ by $\pi(\gamma) = \gamma/\mathcal{B}$. Assume that $\text{Ker}(\pi) \neq 1$ and that for each $B_i \in \mathcal{B}$, there exists $\alpha_i \in \text{Ker}(\pi)$ such that $\alpha|_{B_i}$ is a p^k -cycle, and that there exists $\alpha' \in \text{Ker}(\pi)$ such that $\alpha'|_{B_i} = \alpha|_{B_i}$, and if $i \neq j$, then $\alpha'|_{B_j} = 1$ or $\alpha'|_{B_j}$ is a p^k -cycle. Define an equivalence relation \equiv on \mathcal{B} by $B_i \equiv B_j$ if and only if whenever $\alpha \in \text{Ker}(\pi)$ then $\alpha|_{B_i}$ is a p^k -cycle if and only if $\alpha|_{B_j}$ is also a p^k cycle. Let E_0, E_1, \dots, E_{r-1} be the equivalence classes of \equiv , and $F_i = \cup_{j \in E_i} B_j$. Then $\{F_i : i \in \mathbb{Z}_r\}$ is a complete block system of H and $\alpha|_{F_i} \in H$ for every $\alpha \in \text{Ker}(\pi)$.

Proof. That $\{F_i : i \in \mathbb{Z}_r\}$ is a complete block system of H follows directly from the definition of \equiv . As G is 2-closed and $H \leq G$, we may assume that H is 2-closed. Let Γ be an orbital digraph of H , and let $\alpha \in \text{Ker}(\pi)$. Let $\overrightarrow{ab} \in E(\Gamma)$. Clearly if $a, b \in F_i$ or $a \notin F_i, b \notin F_i$, then $\alpha|_{F_i}(\overrightarrow{ab}) \in E(\Gamma)$. If $a \notin F_i$ but $b \in F_i$, let $a \in B_s, b \in B_t$. Then there exists $\beta \in \text{Ker}(\pi)$ such that $\beta|_{B_s} = 1$ and $\beta|_{B_t}$ is a p^k -cycle or $\beta|_{B_s}$ is a p^k -cycle and $\beta|_{B_t} = 1$. If $\beta|_{B_s} = 1$ and $\beta|_{B_t}$ is a p^k -cycle, then $\overrightarrow{ad} \in E(\Gamma)$ for every $d \in B_t$. As $\text{Ker}(\pi)|_{B_s}$ and $\text{Ker}(\pi)|_{B_t}$ are both transitive on B_s and B_t , respectively, we have that $\overrightarrow{cd} \in E(\Gamma)$ for every $c \in B_s, d \in B_t$. Hence $\alpha|_{F_i}(\overrightarrow{ab}) \in E(\Gamma)$. Similar

arguments will show that if $\beta|_{B_i}$ is a p^k -cycle and $\beta|_{B_i} = 1$, that $\alpha|_{F_i}(\overrightarrow{ab}) \in E(\Gamma)$. Hence $\alpha|_{F_i} \in \text{Aut}(\Gamma)$, and as Γ is an arbitrary orbital digraph of H , that $\alpha|_{F_i} \in H$. If $a \notin F_i$ but $b \in F_i$, then analogous arguments show that $\alpha|_{F_i} \in H$. \square

Theorem 5.9. Let $p \geq 3$ be prime, and $k \geq 1$. Let G act transitively on \mathbb{Z}_{p^k} such that G contains a p^k -cycle and is 2-closed. If Π is a Sylow p -subgroup of G , then Π is a nontrivial wreath product or $\Pi = \Pi_A(\tau_k)$ for some p^k -cycle $\tau_k \in G$ and $A \subset \mathbb{Z}_k^+ \times \mathbb{Z}_k^+$ a nonwreath 2-closed index. Further, Π is 2-closed.

Proof. We first claim that if G is 2-closed, then Π is a nontrivial wreath product or $\Pi = \Pi_A(\tau_k)$. Note that Π admits \mathcal{B}_i , $0 \leq i \leq k$, as a complete block system and that \mathcal{B}_i , $0 \leq i \leq k$ are the only complete block systems of Π . Define π_i as above. If $|\Pi| = p^k$, the $\Pi = \langle \tau_k \rangle = \Pi_\emptyset(\tau_k)$ for some $\tau_k \in \Pi$. Hence we assume that $|\Pi| > p^k$. We will show by induction on i that if Π is not a nontrivial wreath product, then $\text{Ker}(\pi_i) = \langle \tau^{p^i}, P_{\ell,m}(\tau_k) : (\ell, m) \in A_i \rangle$, where $A_i \in \mathbb{Z}_k^+ \times \mathbb{Z}_k^+$ is a nonwreath 2-closed index, and that $\text{Stab}_\Pi(B_{i,0})|_{B_{i,0}} = \text{Ker}(\pi_i)|_{B_{i,0}}$. We will then have that $\Pi = \Pi_{A_1}(\tau_k)$.

As $|\Pi| > p^k$ and p is odd, by Lemma 5.2 $|\text{Ker}(\pi_{k-1})| > p^2$. By Lemma 3.4, there exists $1 \leq j \leq k-1$ such that $\text{Ker}(\pi_{k-1}) = \langle \tau_k^{p^{k-1}}|_B : B \in \mathcal{B}_j \rangle$. If $j = k-1$, then Π is a nontrivial wreath product. Otherwise, let $A_{k-1} = (j, k-1)$. Then $\text{Ker}(\pi_{k-1}) = \langle P_{\ell,m}(\tau_k) : (\ell, m) \in A_{k-1} \rangle$. Clearly we also have that $\text{Stab}_\Pi(B_{k-1,0})|_{B_{k-1,0}} = \text{Ker}(\pi_{k-1})|_{B_{k-1,0}} \cong \mathbb{Z}_p$. Let $i \geq 2$ and $1 \leq i-1 \leq k-1$. Inductively assume that $\text{Ker}(\pi_i) = \langle P_{\ell,m}(\tau_k) : (\ell, m) \in A_i \rangle$ and $\text{Stab}_\Pi(B_{i,0}) = \text{Ker}(\pi_i)|_{B_{i,0}}$.

Define an equivalence relation \equiv on \mathcal{B}_i by $B_{i,\ell} \equiv B_{i,m}$ if and only if whenever $\gamma \in \text{Ker}(\pi_i)$ then $\gamma|_{B_{i,\ell}}$ is a p^{k-i} -cycle if and only if $\gamma|_{B_{i,m}}$ is also a p^{k-i} -cycle. By the induction hypothesis and Lemma 5.1 (ii),

$$\text{Stab}_\Pi(B_{i,0})|_{B_{i,0}} = \text{Ker}(\pi_i)|_{B_{i,0}} = \Pi_{A''}(\tau_k^{p^i}|_{B_{i,0}}). \quad (1)$$

Further, by Lemma 5.1 (iv), if $\gamma \in \text{Ker}(\pi_i)$ such that $\gamma|_{B_{i-1}}|_{B_{i,j}} \neq 1$, then $\gamma|_{B_{i,j}}$ is a p^i -cycle. If $B_{i,j} \not\equiv B_{i,\ell}$, then there exist $\gamma \in \Pi$ such that $\gamma|_{B_{i,j}}$ is a p^{k-i} -cycle, but $\gamma|_{B_{i,\ell}}$ is not. By (1), for fixed ℓ , we may assume that $\gamma|_{B_{i,\ell}} = 1$. Hence by Lemma 5.5, the equivalence classes of \equiv are blocks of Π and if $E_0, E_1, \dots, E_{p^t-1}$ are the equivalence classes of \equiv , then $\tau^{p^{k-i}}|_{E_r} \in \Pi$. We conclude that $\text{Ker}(\pi_i) = \langle \tau^{p^i} P_{\ell,m}(\tau_k) : (\ell, m) \in A_{i-1} \text{ or } (\ell, m) = (i, t-1) \rangle$.

Let $\gamma \in \text{Stab}_\Pi(B_{i,0})$. As each E_j is a block of Π , if $B_{i,0} \in E_0$, then $\gamma(E_0) = E_0$. Further, if $\delta \in \Pi$ such that $\delta(E_0) = E_0$, then by Lemma 5.2, $\text{Stab}_\Pi(E_0)/B_i \cong \mathbb{Z}_{p^r}$ so that if $B_{i,j} \in E_0$, then $\delta(B_{i,j}) = B_{i,j}$. Define $\gamma' : \mathbb{Z}_{p^k} \rightarrow \mathbb{Z}_{p^k}$ by $\gamma'(w) = \gamma(w)$ if $w \notin E_0$ and $\gamma'(w) = w$ if $w \in E_0$. We claim that $\gamma' \in G$, and as $\gamma \in \Pi$, that $\gamma' \in \Pi$. This will imply that $\gamma^{-1}\gamma' \in \text{Ker}(\pi_i)$ so that $\text{Stab}_\Pi(B_{i,0})|_{B_{i,0}} = \text{Ker}(\pi_i)|_{B_{i,0}}$. Hence the claim will follow by induction.

Let Γ be an orbital digraph of G . Then Γ is a circulant digraph of order p^k and $G \leq \text{Aut}(\Gamma)$. We will show that $\gamma' \in \text{Aut}(\Gamma)$. Let $\overrightarrow{ab} \in E(\Gamma)$. Clearly if $\overrightarrow{ab} \in E_0 \times E_0$ or $\overrightarrow{ab} \in (\mathbb{Z}_{p^k} - E_0)^2$, then $\gamma'(\overrightarrow{ab}) \in E(\Gamma)$. If $a \in E_0$ and $b \notin E_0$, then let $a \in B_{i,j}$, $b \in B_{i,\ell}$. Then $\overrightarrow{cd} \in E(\Gamma)$ for every $c \in B_{i,j}$ and $d \in B_{i,\ell}$. Let $\gamma(a) = u$

and $\gamma(b) = v$. As $b \notin E_0$, $v \notin E_0$ and as $a \in B_{i,j}$, $\gamma(a) \in B_{i,j}$. If $v \in B_{i,m}$, then by arguments above, $\vec{cd} \in E(\Gamma)$ for every $c \in B_{i,j}$ and $d \in B_{i,m}$. Hence $\gamma'(\vec{ab}) \in E(\Gamma)$. If $a \notin E_0$ and $b \in E_0$ an analogous argument will show that $\gamma'(\vec{cd}) \in E(\Gamma)$ so that $\gamma' \in \text{Aut}(\Gamma)$. As Γ is an arbitrary orbital digraph of G and G is 2-closed, $\gamma' \in G$.

It now only remains to be shown that Π is 2-closed. If A is a nonwreath 2-closed index, then we will show that there exists a digraph Γ such that $\text{Aut}(\Gamma) = \Pi_A(\tau)$. Let A be a reduced nonwreath 2-closed index. Let $m_1 < m_2 < \dots < m_s \in \mathbb{Z}_k^+$ such that $(\ell_i, m_i) \in A$, $1 \leq i \leq s$, for some $\ell_i \in \mathbb{Z}_k^+$. Let G_i be the unique subgroup of \mathbb{Z}_{p^k} of order p^{k-m_i} , $1 \leq i \leq s$. Let

$$T_{A,k} = \{1 + G_1, p^{\ell_1} + G_2, p^{\ell_2} + G_3, \dots, p^{\ell_{s-1}} + G_s, p^{\ell_s}\}$$

and let $\Gamma_{A,k}$ be the circulant digraph of order p^k with connection set $T_{A,k}$. If $A = \emptyset$, let $T_i = \{1, p^i\}$, $1 \leq i \leq k-1$, and Γ_i be the circulant digraph with connection set T_i . We will show that $\text{Aut}(\Gamma_{A,k}) = \Pi_A(\tau)$ if $A \neq \emptyset$, and if $A = \emptyset$, that $\text{Aut}(\Gamma_i) = \langle \tau \rangle = \Pi_A(\tau)$, for all $1 \leq i \leq k-1$.

If $A = \emptyset$, then any automorphism of Γ_i that is not contained in $\langle \tau \rangle$ must either fix 1 and p^i , or must have $(1 \ p^i)$ in its cycle decomposition. As $p \geq 3$, $\langle \tau \rangle$ is a Sylow p -subgroup of $\text{Aut}(\Gamma_i)$ so that $\text{Aut}(\Gamma_i) \leq N(p^k)$. Hence $|\text{Aut}(\Gamma_i)| \mid (p-1)p^k$. We conclude that $\text{Aut}(\Gamma_i) = \langle \tau \rangle$.

If $A \neq \emptyset$, we proceed by induction on k . If $k = 2$, then we have that $A = \emptyset$. Assume $k-1 \geq 2$ and that for every nonwreath 2-closed index A as above that

$\text{Aut}(\Gamma_{A,k-1}) = \Pi_A(\tau)$. Let A be a minimal nonwreath 2-closed index, $A \subset \mathbb{Z}_k^+ \times \mathbb{Z}_k^+$. If $A = \emptyset$, the result follows with arguments as above. If $A \neq \emptyset$, then consider the graph $\Gamma' = \Gamma_{A,k}/\mathcal{B}_{m_s}$. Now, Γ' is a circulant digraph of order p^{k-m_s} and has connection set

$$T' = \{1 + G_1/G_s, p^{\ell_1} + G_2/G_s, \dots, p^{\ell_{s-2}} + G_{s-1}/G_s, p^{\ell_{s-1}}\}$$

and so by the induction hypothesis, $\text{Aut}(\Gamma') = \Pi_A(\tau/\mathcal{B}_{m_s})$, $A' \subset \mathbb{Z}_{m_s}^+ \times \mathbb{Z}_{m_s}^+$. It is not difficult to see that $\Pi_A(\tau) \leq \text{Aut}(\Gamma_{A,k})$, and that $\text{Ker}(\pi_{m_s}) = P_{\ell_s, m_s}$. We conclude that $\Pi_A(\tau)$ is a Sylow p -subgroup of $\text{Aut}(\Gamma_{A,k})$. As $\Gamma_{A,k} \neq K^{p^k}$ or E^{p^k} , $\text{Aut}(\Gamma_{A,k}) \leq N_{S_{p^k}}(\Pi_A(\tau))$. It is now easily seen that $\text{Aut}(\Gamma_{A,k}) = \Pi_A(\tau)$.

If A is a wreath 2-closed index, we will also show by induction on k that there exists a digraph Γ such that $\text{Aut}(\Gamma) = \Pi_A(\tau_k)$. If $p \geq 3$ and $k \geq 2$, let \vec{C}_p be a directed p -cycle and C_p an undirected cycle. Then $\text{Aut}(C_p) = D_p$, the dihedral group on p -elements, and hence $\text{Aut}(\vec{C}_p) = \mathbb{Z}_p$. By [54], $\text{Aut}(C_p \wr C_p) = D_p \wr D_p$. As \vec{C}_p is a directed cycle, $\text{Aut}(\vec{C}_p \wr \vec{C}_p) = \Pi_{\{1\}}(\tau_2)$. Note that the underlying simple graph of $\vec{C}_p \wr \vec{C}_p$ is not K_{p^2} or E_{p^2} . If $p = 2$, let $\Gamma_1 = E_2$ and Γ_2 be a directed edge. It is then easy to verify that $\text{Aut}(\Gamma) = \Pi_{\{1\}}(\tau_2)$.

Let $k \geq 2$ and inductively assume that for each wreath 2-closed index X there exists a digraph Γ such that $\text{Aut}(\Gamma) = \Pi_A(\tau_k)$. Let $A \subseteq [k+1] \times [k+1]$ be a wreath 2-closed index. Then $\Pi_A(\tau_{k+1}) = \Pi_{A_1}(\tau_i) \wr \Pi_{A_2}(\tau_j)$, $A_1 \subseteq [i] \times [i]$ a 2-closed index and $A_2 \subseteq [j] \times [j]$ a 2-closed index, $i+j = k+1$, $i, j \geq 1$. Then there exists digraphs

Γ_1 and Γ_2 such that $\text{Aut}(\Gamma_1) = \Pi_{A_1}(\tau_i)$ and $\text{Aut}(\Gamma_2) = \Pi_{A_2}(\tau_j)$. It is then easy to verify that $\text{Aut}(\Gamma_1 \wr \Gamma_2) = \Pi_A(\tau_{k+1})$. \square

Corollary 5.10. Let Γ be a circulant digraph of order p^k . Then one of the following is true:

- (i) $\text{Aut}(\Gamma) = S_{p^k}$,
- (ii) $\text{Aut}(\Gamma) = \text{Aut}(\Gamma_i) \wr \text{Aut}(\Gamma_j)$, where $i, j \geq 1$, $i + j = k$, and Γ_i, Γ_j are circulant digraphs of orders p^i and p^j respectively,
- (iii) $\text{Aut}(\Gamma) \leq N_{S_{p^k}}(\Pi_A(\tau))$ for some nonwreath 2-closed index A and $|\text{Aut}(\Gamma)| \mid (p-1)|\Pi_A(\tau)|$.

Proof. If $\text{Aut}(\Gamma)$ is doubly transitive, then $\text{Aut}(\Gamma) = S_{p^k}$. If $\text{Aut}(\Gamma)$ is imprimitive, then the Sylow p -subgroups of $\text{Aut}(\Gamma)$ are 2-closed. If the Sylow p -subgroups of $\text{Aut}(\Gamma)$ are nontrivial wreath products, then so is Γ and (ii) occurs. If not, then by Theorem 5.5 and Lemma 5.3 (iii) holds. \square

Corollary 5.10 gives a recursive algorithm for calculating the automorphism group of a circulant digraph Γ of order p^k , p an odd prime. First one needs to determine what the 2-closed index A of the Sylow p -subgroup Π that contains τ . This can be done without much difficulty by checking whether or not τ raised to an appropriate power and restricted to an appropriate block is contained in $\text{Aut}(\Gamma)$. If the 2-closed index of Π is a nonwreath 2-closed index, then $\text{Aut}(\Gamma) \leq \langle N(p^k), \Pi \rangle$

and can be determined. If not, then $\text{Aut}(\Gamma) = \text{Aut}(\Gamma_1) \wr \text{Aut}(\Gamma_2)$, where Γ_1 is a circulant digraph of order p^i and Γ_2 is a circulant digraph of order p^j , $i + j = k$.

As every 2-closed group can be written as the intersection of automorphism groups of digraphs, we have the following result.

Corollary 5.11. Let G be a 2-closed group that contains τ_k . Then one of the following is true:

- (i) $G = S_{p^k}$,
- (ii) $G = G_1 \wr G_2$, where G_1 and G_2 are 2-closed groups of order p^i and p^j , respectively, such that $i + j = k$ and G_1 and G_2 contain p^i and p^j -cycles, respectively,
- (iii) $G \leq N_{S_{p^k}}(\Pi_A(\tau_k))$ for some nonwreath 2-closed index A and $|G| \mid (p - 1)|\Pi_A(\tau_k)|$.

Corollary 5.12. Let Γ and Γ' be isomorphic circulant graphs of order p^k , $p \geq 3$, with Sylow p -subgroup $\Pi_A(\tau_k)$, with A a reduced 2-closed index. If A is a nonwreath 2-closed index, then there exists $\alpha \in \langle \text{Aut}(\mathbb{Z}_{p^k}), \gamma_{\ell, m} : (\ell, m) \in A \rangle$ such that $\alpha(\Gamma) = \Gamma'$. If A is a wreath 2-closed index, then $\Gamma = \Gamma_1 \wr \Gamma_2$, $\Gamma' = \Gamma'_1 \wr \Gamma'_2$ where Γ_1 and Γ'_1 are isomorphic circulant graphs of order p^i and Γ_2, Γ'_2 are isomorphic circulant graphs of order p^j , where $i + j = k$. Then $\alpha(\Gamma) = \Gamma'$ where $\alpha = \alpha_1 \times \alpha_2$, $\alpha_1(\Gamma_1) = \Gamma'_1$ and $\alpha_2(\Gamma_2) = \Gamma'_2$.

Proof. If A is a nonwreath 2-closed index, then by Theorem 5.5 $\Pi_A(\tau_k) \triangleleft \text{Aut}(\Gamma)$

and $|\text{Aut}(\Gamma)| \mid (p-1)|\Pi_A(\tau_k)|$. It then follows by Lemma 5.3 that there exists $\alpha \in \langle \text{Aut}(\mathbb{Z}_{p^k}), \gamma_{\ell,m} : (\ell, m) \in A \rangle$ such that $\alpha(\Gamma) = \Gamma'$. If A is a wreath 2-closed index, then clearly $\Gamma = \Gamma_1 \wr \Gamma_2$ and $\Gamma' = \Gamma'_1 \wr \Gamma'_2$, where Γ_1 and Γ'_1 are isomorphic circulant graphs of order p^i and Γ_2, Γ'_2 are isomorphic circulant graphs of order p^j , $i + j = k$. It is then easy to verify that if $\alpha_1(\Gamma_1) = \Gamma'_1$ and $\alpha_2(\Gamma_2) = \Gamma'_2$, that $\alpha(\Gamma) = \Gamma'$, where $\alpha = \alpha_1 \times \alpha_2$. \square

CHAPTER 6

ISOMORPHISM PROBLEM FOR THE ELEMENTARY
ABELIAN GROUP OF ORDER A PRIME CUBED

In [5], Babai and Frankl conjectured that \mathbb{Z}_p^k is a CI-group with respect to graphs for all primes p and $k \geq 1$. The case $k = 1$ was settled positively by several authors [1,3,5,6]. It was shown by Godsil [30] that the conjecture is true for $k = 2$. Recently, Nowitz [48] gave an example showing that \mathbb{Z}_2^k is not a CI-group with respect to graphs for all $k \geq 6$, and asked if there existed a prime p_0 so that if $p \geq p_0$ and p is prime, then \mathbb{Z}_p^3 is not a CI-group with respect to graphs. We will answer this question negatively by showing that \mathbb{Z}_p^3 is a CI-group with respect to graphs for all primes p . We first prove a lemma that settles the case when Γ is a wreath product of two graphs.

Lemma 6.1. If Γ is a Cayley graph of \mathbb{Z}_p^3 and Γ is isomorphic to a Cayley graph of \mathbb{Z}_p^3 that is the wreath product of a circulant graph of order p over a Cayley graph of \mathbb{Z}_p^2 or the wreath product of a Cayley graph of \mathbb{Z}_p^2 over a circulant graph of order p , then Γ is a CI-graph with respect to \mathbb{Z}_p^3 .

Proof. We will show that if Γ is a Cayley graph of \mathbb{Z}_p^3 and Γ is isomorphic to a wreath product of a circulant graph Γ_1 of order p over a Cayley graph Γ_2 of \mathbb{Z}_p^2 , then Γ is a CI-graph. The other case follows with a similar argument.

Let $\Gamma \cong \Gamma_1 \wr \Gamma_2$ be as above. Let Γ' be a Cayley graph of \mathbb{Z}_p^3 such that Γ' is isomorphic to Γ . Let $\tau_1, \tau_2, \tau_3 : \mathbb{Z}_p^3 \rightarrow \mathbb{Z}_p^3$ by $\tau_1(i, j, k) = (i + 1, j, k)$, $\tau_2(i, j, k) =$

$(i, j + 1, k)$, and $\tau_3(i, j, k) = (i, j, k + 1)$. Then $G = \langle \tau_1, \tau_2, \tau_3 \rangle \leq \text{Aut}(\Gamma)$, $G \leq \text{Aut}(\Gamma')$, and $G \cong \mathbb{Z}_p^3$. Let Π be a Sylow p -subgroup of $\text{Aut}(\Gamma)$ that contains G . Then Π has nontrivial center so there exists $\alpha \in C(\Pi)$, the center of Π , $\alpha \neq 1$. As $\alpha \in C(\Pi)$, $\alpha \in C_{S_{\mathbb{Z}_p^3}}(G)$, the centralizer of G in $S_{\mathbb{Z}_p^3}$, and as G is regular and abelian, $\alpha \in G$. Now, Π admits a complete block system \mathcal{B} of p^2 blocks of size p , where the blocks of size p are formed by the orbits of α . Define $\pi_1 : \Pi \rightarrow S_{\mathcal{B}}$ by $\pi_1(\gamma) = \gamma/\mathcal{B}$. Then Π/\mathcal{B} is a p -group and there exists $\beta \in \Pi$ such that $\beta/\mathcal{B} \in C(\Pi/\mathcal{B})$, $\beta/\mathcal{B} \neq 1$. As $G/\mathcal{B} \leq \Pi/\mathcal{B}$, $\beta/\mathcal{B} \in G/\mathcal{B}$, so that $\beta = \beta'\omega$, $\beta' \in G$, $\omega \in \text{Ker}(\pi_1)$. Hence we may assume without loss of generality that $\beta \in G$. Then Π admits a complete block system \mathcal{C} of p blocks of size p^2 , where the elements of \mathcal{C} are formed by the orbits of $\langle \alpha, \beta \rangle$. Define $\pi_2 : \Pi \rightarrow S_p$ by $\pi_2(\gamma) = \gamma/\mathcal{C}$. Then $|\Pi/\mathcal{C}| = p$, and if $\gamma \in \Pi$ such that $\gamma/\mathcal{C} \neq 1$, then $\gamma = \gamma'\omega'$, $\gamma' \in G$, $\omega' \in \text{Ker}(\pi_2)$. Thus we assume that $\gamma \in G$. Hence $\langle \alpha, \beta, \gamma \rangle = \langle \tau_3, \tau_2, \tau_1 \rangle$ and so by [19] there exists $\delta_1 \in \text{Aut}(\mathbb{Z}_p^3)$ so that $\delta_1^{-1}\alpha\delta_1 = \tau_3$, $\delta_1^{-1}\beta\delta_1 = \tau_2$, and $\delta_1^{-1}\gamma\delta_1 = \tau_1$. Further, Γ is a CI-graph if and only if $\delta_1(\Gamma)$ is a CI-graph, and as $\langle \alpha|_{\mathcal{C}}, \beta|_{\mathcal{C}} : \mathcal{C} \in \mathcal{C} \rangle \leq \Pi$, $\delta_1(\Gamma)$ is the wreath product of an order p -circulant Γ'_1 over a Cayley graph Γ'_2 of \mathbb{Z}_p^2 . As \mathbb{Z}_p^2 is a CI-group with respect to graphs [30], clearly there exists $\delta_2 \in \text{Aut}(\mathbb{Z}_p^3)$ such that $\delta_2\delta_1(\Gamma) = \Gamma_1 \wr \Gamma_2$. By analogous arguments, there exists $\delta'_1, \delta'_2 \in \text{Aut}(\mathbb{Z}_p^3)$ so that $\delta'_2\delta'_1(\Gamma') = \Gamma_1 \wr \Gamma_2$ so that $\delta'^{-1}_1\delta'^{-1}_2\delta_2\delta_1(\Gamma) = \Gamma'$. Hence Γ is a CI-graph for \mathbb{Z}_p^3 . \square

Theorem 6.2. \mathbb{Z}_p^3 is a CI-group with respect to graphs.

Proof. Let Γ and Γ' be isomorphic Cayley graphs for \mathbb{Z}_p^3 , and $\varphi : \Gamma \rightarrow \Gamma'$ an isomorphism. Let τ_1, τ_2, τ_3 and G be as in Lemma 3.4. We must show that there exists $\delta \in \text{Aut}(\mathbb{Z}_p^3)$ such that $\delta(\Gamma) = \Gamma'$ or that $\varphi^{-1}G\varphi$ and G are conjugate in $\text{Aut}(\Gamma)$. Now, G and $\varphi^{-1}G\varphi$ are contained in Sylow p -subgroups Π and Π' , respectively, of $\text{Aut}(\Gamma)$, and so there exists $\gamma \in \text{Aut}(\Gamma)$ such that $\gamma^{-1}\varphi^{-1}G\varphi\gamma \leq \Pi$. As Π is a p -group, there exists $\alpha \in C(\Pi)$, $\alpha \neq 1$, and by arguments in Lemma 6.1, we may assume $\alpha \in G$. Hence Π admits a complete block system \mathcal{B} of p^2 blocks of size p , where the elements of \mathcal{B} are the orbits of α . Define $\pi_1 : \Pi \rightarrow S_{\mathcal{B}}$ by $\pi_1(\gamma) = \gamma/\mathcal{B}$. By Lemma 3.4, $|\text{Ker}(\pi_1)| = p, p^p$, or p^{p^2} . If $|\text{Ker}(\pi_1)| = p^{p^2}$, then $\text{Ker}(\pi_1) = \langle \alpha|_B : B \in \mathcal{B} \rangle$ and Γ is isomorphic to the wreath product of a Cayley graph of \mathbb{Z}_p^2 over an order p -circulant. Thus by Lemma 3.4, there exists $\delta \in \text{Aut}(\mathbb{Z}_p^3)$ such that $\delta(\Gamma) = \Gamma'$. We therefore assume that $|\text{Ker}(\pi_1)| = p$ or p^p .

If $|\text{Ker}(\pi_1)| = p^p$, then by Lemma 3.4 Π admits a complete block system \mathcal{C} of p blocks of size p^2 , where if $C \in \mathcal{C}$, then there exists $\alpha_C \in \text{Ker}(\pi_1)$ such that $\alpha_C(c) \neq c$ for all $c \in C$ and $\alpha_C(d) = d$ for all $d \in \mathbb{Z}_p^3 - C$. Define $\pi_2 : \Pi \rightarrow S_p$ by $\pi_2(\gamma) = \gamma/\mathcal{C}$. Clearly \mathcal{C} is also a complete block system for G , and $|G/\mathcal{C}| = p$. Hence there exists $\beta \in G$ such that $\beta/\mathcal{C} = 1$ but $\beta \notin \langle \alpha \rangle$. Thus \mathcal{C} is formed by the orbits of $\langle \alpha, \beta \rangle$. Further, $\Pi/\mathcal{C} = G/\mathcal{C}$ so there exists $\gamma \in \Pi$ such that γ/\mathcal{C} is semiregular, $\gamma/\mathcal{C} \in G/\mathcal{C}$. By arguments above, we assume $\gamma \in G$. Then $\langle \alpha, \beta, \gamma \rangle = G$ and by arguments in Lemma 6.1 we may assume that $\alpha = \tau_3$, $\beta = \tau_2$, and $\gamma = \tau_1$.

Now, $|\Pi/\mathcal{B}| = p^2$ or $|\Pi/\mathcal{B}| > p^2$. If $|\Pi/\mathcal{B}| > p^2$, then as the elements of \mathcal{C} are formed by the orbits of $\langle \tau_2, \tau_3 \rangle$, $\tau_2(C) = C$ for all $C \in \mathcal{C}$. Hence $\text{Ker}(\pi_1) =$

$\langle \tau_3|_C : C \in \mathcal{C} \rangle$. Let $C_i = \{(i, j, k) : j, k \in \mathbb{Z}_p\}$ and $B_{i,j} = \{(i, j, k) : k \in \mathbb{Z}_p\}$. Then $\mathcal{C} = \{C_i : i \in \mathbb{Z}_p\}$ and $\mathcal{B} = \{B_{i,j} : i, j \in \mathbb{Z}_p\}$. Suppose that some vertex of $B_{i,a}$ is adjacent to some vertex of $B_{j,b}$, $i \neq j$. Then every vertex of $B_{i,a}$ is adjacent to every vertex of $B_{j,b}$. As $|\Pi/\mathcal{B}| > p^2$, there exists $\beta \in \Pi$ such that $\beta|_{C_c}/\mathcal{B} = 1$ and $\beta|_{C_d}/\mathcal{B} \neq 1$, $c \neq d$. As p is prime, we may assume that $c - d \equiv i - j \pmod{p}$, and by conjugating by τ_1 , if necessary, that $c = i$ and $d = j$. As $\beta|_{C_i}/\mathcal{B} \neq 1$, $\beta|_{C_i}/\mathcal{B}$ is a p -cycle on the blocks $\{B_{i,k} : k \in \mathbb{Z}_p\}$, and as $\beta|_{C_j}/\mathcal{B} = 1$, β fixes each block $B_{j,k}$, $k \in \mathbb{Z}_p$. Thus every vertex of $B_{i,a}$ is adjacent to every vertex of C_j , and by symmetry, every vertex of C_i is adjacent to every vertex of C_j . We conclude that Γ is the wreath product of an order p -circulant over a Cayley graph of \mathbb{Z}_p^2 , and so Γ is a CI-graph for \mathbb{Z}_p^3 .

If $|\Pi/\mathcal{B}| = p^2$, then $\text{Ker}(\pi_1) = \langle \tau_3|_C : C \in \mathcal{C} \rangle$, and so if $\varphi_1 = \varphi\gamma$, then $\varphi_1^{-1}\mathcal{C}\varphi_1 = \mathcal{C}$. Hence $\varphi_1(i, j, k) = (\sigma(i), \xi_i(j, k))$, $\sigma \in S_p$, $\xi_i \in S_{\mathbb{Z}_p^2}$. As $\text{Ker}(\pi_2)|_C \cong \mathbb{Z}_p^2$ for all $C \in \mathcal{C}$, $\xi_i(j, k) = \omega_i(j, k) + (a_i, b_i)$, $\omega_i \in \text{Aut}(\mathbb{Z}_p^2)$, $a_i, b_i \in \mathbb{Z}_p$. As $\omega_i \in \text{Aut}(\mathbb{Z}_p^2)$,

$$\omega_i(j, k) = (\alpha_i j + \beta_i k, \gamma_i k + \iota_i j),$$

$\alpha_i, \beta_i, \gamma_i, \iota_i \in \mathbb{Z}_p$, where the 2×2 matrix with first row $\alpha_i \ \beta_i$ and second row $\gamma_i \ \iota_i$ has non-zero determinant. If $\beta_i \neq 0$ for any i , then, as $\text{Ker}(\pi_1) = \langle \tau_3|_C : C \in \mathcal{C} \rangle$, $|\Pi/\mathcal{B}| > p^2$, so $\beta_i = 0$ for all $i \in \mathbb{Z}_p$. As $\text{Aut}(\Gamma') = \varphi_1^{-1}\text{Aut}(\Gamma)\varphi_1$, we conclude that $\text{Ker}(\pi_1) \leq \text{Aut}(\Gamma')$ and so we may assume (by right multiplication by elements of $\text{Ker}(\pi_1)$) that $b_i = 0$ for all $i \in \mathbb{Z}_p$. We now show that $\alpha_i = \alpha_j$ for all $i, j \in \mathbb{Z}_p$.

As $|\Pi/C| = p$, $\sigma(i) = ri + c$, $r \in \mathbb{Z}_p^*$, $c \in \mathbb{Z}_p$, and as $\tau_1 \in \text{Aut}(\Gamma')$, we may assume that $\sigma(i) = ri$. Hence

$$\varphi_1(i, j, k) = (ri, \alpha_i j + a_i, \gamma_i k + \iota_i j),$$

and so

$$\varphi_1^{-1}(i, j, k) = (r^{-1}i, \alpha_{r^{-1}i}^{-1}(j - a_{r^{-1}i}), \gamma_{r^{-1}i}^{-1}k - \gamma_{r^{-1}i}^{-1}\iota_{r^{-1}i}j).$$

Hence if $\tau = \tau_1^{-r^{-1}} \varphi_1^{-1} \tau_1 \varphi_1$, then $\tau \in \text{Ker}(\pi_2)$ and

$$\tau(i, j, k) = (i, \alpha_{i+r^{-1}}^{-1} \alpha_i j + c_i, \theta_i(j, k)),$$

for some $c_i \in \mathbb{Z}_p$ and $\theta_i : \mathbb{Z}_p^2 \rightarrow \mathbb{Z}_p$. Now, $|\tau| = p^t$, $t \geq 0$, and $|\mathbb{Z}_p^*| = p - 1$, so that $\alpha_{i+r^{-1}}^{-1} \alpha_i = 1$. Hence $\alpha_i = \alpha_{i+r^{-1}}$, and as $\langle r^{-1} \rangle = \mathbb{Z}_p$, $\alpha_i = \alpha_j$ for all $i, j \in \mathbb{Z}_p$.

Let $\alpha = \alpha_0$. Then

$$\tau(i, j, k) = (i, j + \alpha^{-1}(a_i - a_{i+r^{-1}}), \theta_i(j, k)).$$

As $|\Pi/B| = p^2$, $\alpha^{-1}(a_i - a_{i+r^{-1}}) = c$, $c \in \mathbb{Z}_p$, so $a_{i+r^{-1}} = a_i - \alpha c$. As $\tau_2 \in \text{Aut}(\Gamma')$, we may assume that $a_0 = 0$ and so $a_{ir^{-1}} = -i\alpha c$. Hence $a_i = -ir\alpha c$. Define $\phi : \mathbb{Z}_p^3 \rightarrow \mathbb{Z}_p^3$ by $\phi(i, j, k) = (i, j - ir\alpha c, k)$. Then $\phi \in \text{Aut}(\mathbb{Z}_p^3)$, and if $\varphi'_1 = \varphi_1 \phi$, we may assume without loss of generality (by replacing Γ' by $\phi^{-1}(\Gamma')$) that $c = 0$ and $\varphi_1 = \varphi'_1$. Hence $a_i = a_{i+r}$, and as $\langle r^{-1} \rangle = \mathbb{Z}_p$,

$$\varphi_1(i, j, k) = (ri, \alpha j + a, \gamma_i k + \iota_i j).$$

As $\tau_2 \in \text{Aut}(\Gamma')$, we may assume that $a = 0$. Now, elementary calculations will show that

$$\theta_i(j, k) = \gamma_{i+r-1}^{-1} \gamma_i k + \gamma_{i+r-1}^{-1} (\iota_i - \iota_{i+r-1}) j.$$

Further, $\tau \in \text{Ker}(\pi_1)$ and so $\theta_i \in \langle \tau_3|_C : C \in \mathcal{C} \rangle$. Thus $\theta_i(j, k) = k + c_i$, $c_i \in \mathbb{Z}_p$.

Hence for $k = 0$,

$$(\iota_i - \iota_{i+r-1})j = \gamma_{i+r-1} c_i$$

for all $j \in \mathbb{Z}_p$. We conclude that $\iota_i = \iota_{i+r-1}$ for all $i \in \mathbb{Z}_p$, and so $\iota_i = \iota_j$ for all $i, j \in \mathbb{Z}_p$. Let $\iota = \iota_0$. Then $\theta_i(j, k) = \gamma_{i+r-1}^{-1} \gamma_i k$, and as $|\theta_i| = p$, $\gamma_{i+r-1}^{-1} \gamma_i = 1$ for all $i \in \mathbb{Z}_p$. Hence $\gamma_i = \gamma_j$ for all $i, j \in \mathbb{Z}_p$. Thus if $\gamma = \gamma_0$, then

$$\varphi_1(i, j, k) = (ri, \alpha j, \gamma k + \iota j),$$

and so $\varphi_1 \in \text{Aut}(\mathbb{Z}_p^3)$. Hence Γ and Γ' are isomorphic by $\varphi_1 \in \text{Aut}(\mathbb{Z}_p^3)$ and so Γ is a CI-graph.

If $|\text{Ker}(\pi_1)| = p$, then $|\Pi/\mathcal{B}| = p^2$ or $|\Pi/\mathcal{B}| > p^2$. If $|\Pi/\mathcal{B}| = p^2$, then $|\Pi| = p^3$ so that G and $\varphi^{-1}G\varphi$ are conjugate in $\text{Aut}(\Gamma)$ and so Γ is a CI-graph. If $|\Pi/\mathcal{B}| > p^2$, by arguments in Lemma 6.1, there exists $\beta, \gamma \in G$ so that $\langle \alpha, \beta, \gamma \rangle = G$, and Π

admits a complete block system \mathcal{C} of p blocks of size p^2 , where the elements of \mathcal{C} are formed by the orbits of $\langle \alpha, \beta \rangle$. Also by arguments in Lemma 6.1, we assume without loss of generality that $\alpha = \tau_3$, $\beta = \tau_2$, and $\gamma = \tau_1$.

If $\text{Ker}(\pi_2)|_{\mathcal{C}} = \langle \tau_2, \tau_3 \rangle|_{\mathcal{C}}$, then if $\omega \in \text{Ker}(\pi_2)$, $\omega(i, j, k) = (i, j + a_i, k + b_i)$, $a_i, b_i \in \mathbb{Z}_p$. Thus if $\omega \in \Pi$, $\omega(i, j, k) = (i + s, j + a_i, k + b_i)$, $s \in \mathbb{Z}_p$, and so

$$\gamma\tau_2(i, j, k) = (i + s, j + 1 + a_i, k + b_i) = \tau_2\gamma(i, j, k).$$

Hence $\tau_2 \in C(\Pi)$, and Π admits a complete block system \mathcal{B}_ρ of p^2 blocks of size p , where \mathcal{B}_ρ is formed by the orbits of $\rho \in \langle \tau_2, \tau_3 \rangle$. Define $\pi_\rho : \Pi \rightarrow S_{\mathbb{Z}_p^3/\mathcal{B}_\rho}$ by $\pi_\rho(\gamma) = \gamma/\mathcal{B}_\rho$. If $|\text{Ker}(\pi_\rho)| > p$ for any $\rho \in \langle \tau_2, \tau_3 \rangle$, then by arguments above Γ is a CI-object for \mathbb{Z}_p^3 . We now show that such a ρ always exists.

As $|\text{Ker}(\pi_1)| = p$, Γ is not isomorphic to a wreath product of a circulant graph of order p over a Cayley graph for \mathbb{Z}_p^2 . Let $\alpha \in \Pi$ such that $\alpha/\mathcal{B} \neq 1$ but α/\mathcal{B} fixes some block $B \in \mathcal{B}$. Such a α exists as $|\Pi/\mathcal{B}| > p^2$. Without loss of generality, assume that $\alpha(B_{0,0}) = B_{0,0}$. Then $\alpha|_{B_{0,0}} \in \langle \tau_3|_{B_{0,0}} \rangle$. Hence there exists $s \in \mathbb{Z}_p$ such that $\alpha\tau_3^s(0, 0, 0) = (0, 0, 0)$, so we assume that $\alpha(0, 0, 0) = (0, 0, 0)$. Hence $\alpha(0, j, k) = (0, j, k)$ for all $j, k \in \mathbb{Z}_p$. Let T be the connection set of Γ . As Γ is not isomorphic to a wreath product of an order p -circulant over a Cayley graph of \mathbb{Z}_p^2 , there exists $i \in \mathbb{Z}_p$ such that $C_i \cap T \neq \emptyset$ but $C_i \not\subseteq T$. Further, as $\alpha/\mathcal{B} \neq 1$, there exists $j \in \mathbb{Z}_p$ such that $\alpha|_{C_{i,j}} = 1$ but $\alpha|_{C_{i(j+1)}} \neq 1$. Let $\rho \in \langle \tau_2, \tau_3 \rangle$ such that $\alpha|_{C_{i(j+1)}} = \rho|_{C_{i(j+1)}}$, and denote the orbits of $\rho|_{C_i}$ by $\mathcal{O}_0, \mathcal{O}_1, \dots, \mathcal{O}_{p-1}$.

Then if $(0, 0, 0)$ is adjacent to $(i, j, k) \in \mathcal{O}_\ell$, then $(0, 0, 0)$ is adjacent to every vertex of \mathcal{O}_ℓ . Observe that if $\tau \in \langle \tau_2, \tau_3 \rangle$ such that $\tau \notin \langle \rho \rangle$, then each orbit of $\tau|_{C_k}$ contains exactly one element of each orbit of $\rho|_{C_k}$ for all $k \in \mathbb{Z}_p$. We conclude that $\alpha|_{C_{i(j+2)}} \in \langle \rho \rangle|_{C_{i(j+2)}}$, or $C_i \subseteq T$. As $C_i \not\subseteq T$, $\alpha|_{C_{i(j+2)}} \in \langle \rho \rangle|_{C_{i(j+2)}}$. Arguing similarly, we have that $\alpha|_{C_{i(j+3)}} \in \langle \rho \rangle|_{C_{i(j+3)}}$. Continuing in this fashion, we have that $\alpha|_{C_{ik}} \in \langle \rho \rangle|_{C_{ik}}$ for all $k \in \mathbb{Z}_p$, and so that $\alpha/B_\rho = 1$. As $\alpha \neq \rho$, $|\text{Ker}(\pi_\rho)| > p$.

If $\text{Ker}(\pi_2)|_C \neq \langle \tau_2, \tau_3 \rangle|_C$, let $\alpha \in \text{Ker}(\pi_2)$ such that $\alpha|_C \notin \langle \tau_2, \tau_3 \rangle$. Consider $\alpha^{-1}\tau_2\alpha$. As $\Pi/B \leq S_{\mathbb{Z}_p^2}$, Π/B is contained in a Sylow p -subgroup of $S_{\mathbb{Z}_p^2}$, which is isomorphic to $C_p \wr C_p$, where C_p is a cyclic group of order p . Hence $\langle \tau_2, \alpha \rangle/B \leq 1_{S_p} \wr C_p$. As $1_{S_p} \wr C_p$ is an abelian group, $\alpha^{-1}\tau_2\alpha/B = \tau_2/B$. Thus $\alpha^{-1}\tau_2\alpha\tau_2^{-1} \in \text{Ker}(\pi_2)$ and so $\alpha^{-1}\tau_2\alpha = \tau_2\tau_1^a$, $a \in \mathbb{Z}_p$. We conclude that $\alpha(i, j, k) = (i, \theta_i(j, k))$, where

$$\theta_i(j, k) = \omega_i(j, k) + (a_i, b_i),$$

$\omega_i \in \text{Aut}(\mathbb{Z}_p^2)$, $a_i, b_i \in \mathbb{Z}_p$. Let $\beta_i : \mathbb{Z}_p^2 \rightarrow \mathbb{Z}_p^2$ by $\beta_i(j, k) = (j + a_i, k + b_i)$. Then $\theta_i = \beta_i\omega_i$. Let $\omega, \beta : \mathbb{Z}_p^3 \rightarrow \mathbb{Z}_p^3$ by $\omega(i, j, k) = (i, \omega_i(j, k))$ and $\beta(i, j, k) = (i, \beta_i(j, k))$. Then $\alpha = \beta\omega$ and so

$$\alpha^{-1}\tau_2\alpha = \omega^{-1}\beta^{-1}\tau_2\beta\omega = \omega^{-1}\tau_2\omega = \tau_2\tau_1^a,$$

where $a \in \mathbb{Z}_p$. Hence $\omega_i = \omega_j$ for all $i, j \in \mathbb{Z}_p$. Without loss of generality assume that $a_0 = 0$ and $b_0 = 0$. We will consider when $\alpha \in \text{Aut}(\mathbb{Z}_p^3)$ and when $\alpha \notin \text{Aut}(\mathbb{Z}_p^3)$.

If $\alpha \notin \text{Aut}(\mathbb{Z}_p^3)$, then $\alpha^{-1}\tau_1\alpha \notin \langle \tau_1, \tau_2, \tau_3 \rangle$. Further, note that

$$\alpha_1(i, j, k) = \tau_1^{-1}\alpha^{-1}\tau_1\alpha(i, j, k) = (i, (j, k) + \omega^{-1}((a_i, b_i) - (a_{i+1}, b_{i+1}))). \quad (1)$$

As $\alpha \notin \text{Aut}(\mathbb{Z}_p^3)$, $\alpha_1 \notin \langle \tau_1, \tau_2, \tau_3 \rangle$. Let $H = \langle \tau_1, \tau_2, \tau_3, \alpha_1 \rangle$. Note that \mathcal{B} and \mathcal{C} are still complete block systems for $H \leq \Pi$. Define $\pi'_1 : H \rightarrow S_{\mathbb{Z}_p}$ by $\pi'_1(\delta) = \delta/\mathcal{B}$ and $\pi'_2 : H \rightarrow S_p$ by $\pi'_2(\delta) = \delta/\mathcal{C}$. Then $\text{Ker}(\pi'_1) \leq \text{Ker}(\pi_1) = \langle \tau_3 \rangle$ so that $\text{Ker}(\pi'_1) = \langle \tau_3 \rangle$. As $|H| \geq p^4$ and $|\text{Im}(\pi'_2)| = p$, $|\text{Ker}(\pi'_2)| \geq p^3$. By (1), $\text{Ker}(\pi'_2)|_{\mathcal{C}} \leq \langle \tau_2, \tau_3 \rangle|_{\mathcal{C}}$ for all $C \in \mathcal{C}$, and so by arguments above there exists $\rho \in H \cap \langle \tau_2, \tau_3 \rangle$ such that if $\pi'_\rho : H \rightarrow S_{\mathbb{Z}_p}$ by $\pi'_\rho(\delta) = \delta/\mathcal{B}_\rho$ (\mathcal{B}_ρ being the orbits of ρ), then $|\text{Ker}(\pi'_\rho)| > p$. By Lemma 3.4, $|\text{Ker}(\pi'_\rho)| = p^{p^2+1}$ or p^{p+1} . If $|\text{Ker}(\pi'_\rho)| = p^{p^2+1}$, then Γ is isomorphic to the wreath product of a Cayley graph of $\mathbb{Z}_p \times \mathbb{Z}_p$ over an order p -circulant, and so by Lemma 6.1, Γ is a CI-graph. If $|\text{Ker}(\pi'_\rho)| = p^{p+1}$, then by Lemma 3.4 $\langle \rho|_{\mathcal{C}} : C \in \mathcal{C} \rangle \leq \text{Aut}(\Gamma)$. Further, $\rho \in \langle \tau_2, \tau_3 \rangle$ and $\rho \notin \langle \tau_3 \rangle$, so that $\rho = \tau_2^b \tau_1^c$, $b, c \in \mathbb{Z}_p$, $a \neq 0$. Thus ρ permutes the blocks of \mathcal{B} as a p -cycle.

Now, $(\alpha|_{\mathcal{C}_i})/\mathcal{B} \in \langle \tau_2|_{\mathcal{C}_i} \rangle/\mathcal{B}$ for all $i \in \mathbb{Z}_p$. Let $d_i \in \mathbb{Z}_p$ such that $(\alpha|_{\mathcal{C}_i})/\mathcal{B} = \tau_2^{d_i}/\mathcal{B}$. Let $f_0, f_1, \dots, f_{p-1} \in \mathbb{Z}_p$ such that $f_i a = d_i$. Then

$$(\alpha \prod_{i=0}^{p-1} \rho^{-f_i} |_{\mathcal{C}_i})/\mathcal{B} = 1.$$

Let $\alpha' = \alpha \prod_{i=0}^{p-1} \rho^{-f_i}$. As $\alpha|_{\mathcal{C}} \notin \langle \tau_2, \tau_3 \rangle|_{\mathcal{C}}$ for some $C \in \mathcal{C}$ and $\rho^{-f_i}|_{\mathcal{C}} \in \langle \tau_2, \tau_3 \rangle$ for all i , $\alpha'|_{\mathcal{C}} \notin \langle \tau_2, \tau_3 \rangle$ and thus $\alpha' \notin \langle \tau_3 \rangle$ but $\alpha' \in \text{Ker}(\pi_1)$, a contradiction. Hence $\alpha \in \text{Aut}(\mathbb{Z}_p^3)$.

If $\alpha \in \text{Aut}(\mathbb{Z}_p^3)$, then $\Pi \leq \text{AGL}_3(p)$, the affine group over the field with p^3 elements. As is well known, this group is doubly transitive and by [58], Theorem 11.5, $G = \langle \tau_1, \tau_2, \tau_3 \rangle$ is the only minimal normal subgroup of $\text{AGL}_3(p)$, and also of Π . Thus $\varphi_1^{-1}G\varphi_1 = G$ and Γ is a CI-graph of \mathbb{Z}_p^3 . \square

CHAPTER 7

PROBLEMS IN EXTREMAL GRAPH THEORY

In 1963 Erdős and Sós [27] conjectured that every graph with n vertices and size $\lfloor (k-1)n/2 \rfloor + 1$ contains every tree of order $k+1$. Since then, to the best knowledge of the author, no results directly pertaining to this conjecture have been proven, although Erdős and Gallai [26] had proved in 1959 that every such graph contains a path of length k . This fact follows from an application of a celebrated result of Dirac [21], and the simple observation that every such graph contains an induced subgraph H such that if $|V(H)| = m$, then $|E(H)| \geq \lfloor (k-1)m/2 \rfloor + 1$ with minimal degree $\lceil k/2 \rceil$. As every graph that satisfies the hypothesis of the Erdős-Sós conjecture also contains a vertex of degree k , it is clear that the conjecture is true for every such graph with girth at least k . It thus seems reasonable to attempt to verify the conjecture for graphs with girth smaller than k to obtain partial results.

Our main result is the following theorem.

Theorem 9.2. Let G be a graph with girth $g \geq 5$ and $|E(G)| \geq \lfloor (k-1)n/2 \rfloor + 1$. Then G contains every tree of order $k+1$.

This result is an easy consequence of the following result.

Theorem 9.1. Let G be a graph with girth $g \geq 2t+1$, $t \geq 2$, and $k \geq 5$. If $\delta(G) \geq \lceil k/2 \rceil$ then G contains every tree T such that $m = \Delta(T) \leq \lceil k/2 \rceil$ and the

order of G is $\lfloor \ell k \rfloor + 1$, where

$$\ell \leq \frac{t}{2} - \frac{(t-2)(m-1)}{k}.$$

Furthermore, if $\Delta(G) \geq d_1 > \lceil k/2 \rceil$, then G contains every tree of order $k+1$ with degree sequence $(d_1, d_2, \dots, d_{k+1})$, $d_i \geq d_{i+1}$.

Theorem 9.1 not only allows us to prove Theorem 9.2, but allows us to give a characterization of the extremal graphs.

Corollary 9.3. If G is a graph with girth $g \geq 5$ and $|E(G)| = \lfloor (k-1)n/2 \rfloor$, then if k is odd G does not contain every tree of order $k+1$ if and only if G is $(k-1)$ -regular, and if k is even, then G does not contain every tree of order $k+1$ if and only if at most one vertex of G has degree $k-2$ and every other vertex has degree $k-1$.

The special case of Theorem 2 where $t = 2$ is likely to generalize to a consistently wider class of graphs.

Conjecture 9.4. Let G be a graph with girth $g \geq 2t+1$ and $\delta(G) \geq k \geq 3$. Then G contains every tree T of order $kt+1$ such that $\Delta(T) \leq k$.

Finally, we will prove Conjecture 9.4 if $m = \Delta(T)$ is fixed, $g \geq 9$, and k is sufficiently large.

Theorem 10.5. Let $\epsilon > 0$, $g \geq 9$, and $n \geq 1$. Then there exists k' such that if $k \geq k'$ then every graph G with girth g and $\delta(G) = k$ contains every tree T of order $nk + 1$ such that $\Delta(T) \leq (1 - \epsilon)k$.

The next problem that we will consider is the Tree Packing Conjecture. Let G_1, \dots, G_ℓ , and G be graphs. We say that G_1, \dots, G_ℓ can be packed into G if there exists inclusions $V(G_i) \subseteq V(G)$, $1 \leq i \leq \ell$, such that if $e \in E(G_i)$, then $e \notin E(G) - \cup_{j \neq i} E(G_j)$. The Erdős-Sós conjecture motivated Gyárfás and Lehel [32] to make the following conjecture, usually referred to as the Tree Packing Conjecture.

Conjecture 7.1. Any sequence of trees T_1, \dots, T_n such that $|V(T_i)| = i$ can be packed into K^n .

Many special cases of the Tree Packing Conjecture have been solved (see [60]), the most significant being those of Fishburn [29]. Fishburn was primarily interested in two conjectures of his that jointly imply the Tree Packing Conjecture. Let H_{2n} be the unique graph (upto isomorphism) on $2n$ vertices with degree sequence $(2n - 1, 2n - 2, \dots, n + 1, n, n, n - 1, \dots, 2, 1)$. Let H_{2n+1} be the unique (upto isomorphism) graph on $2n + 1$ vertices with degree sequence $(2n, 2n - 1, \dots, n + 1, n, n, n - 1, \dots, 2, 1)$. The graph H_n is referred to as the *half-complete graph*. Fishburn made the two following conjectures.

Conjecture 7.2. For n even, every sequence of tree T_1, T_3, \dots, T_{n-1} , such that $|V(T_i)| = i$, can be packed into the half-complete graph H_n .

Conjecture 7.3. For n odd, every sequence of tree T_2, T_4, \dots, T_{n-1} , such that $|V(T_i)| = i$, can be packed into the half-complete graph H_n .

Fishburn proved that if T_{n+1} is one of several special types of trees on $n + 1$ vertices, then H_n and T_{n+1} pack into H_{n+2} . This implies the Tree Packing Conjecture for those special types of trees. Also, he verified the Tree Packing Conjecture for $k \leq 9$. However, he also gave examples of trees T_{n+1} that do not pack with H_n into H_{n+2} . Furthermore, Balakrishnan, Basha, and Paulraja [6] gave the following necessary condition for T_{n+1} and H_n to pack into H_{n+2} .

Theorem 8.4. Let T_{n+1} be packable with H_n into H_{n+2} and let $v_0 \dots v_m v_{m+1}$ be the longest path in T_{n+1} . Then for every vertex v of T_{n+1} not on P , the distance between v and P in T_{n+1} is at most 2.

Bhat-Nayak and Sethuraman [11] have generalized this necessary condition. The main idea behind this discussion, is that Fishburn's techniques will only allow sequences of trees T_1, \dots, T_n that are, in some sense, 'almost' paths to be packed into K_n . We take, somewhat, the opposite approach, and wish to pack 'almost' stars. For a graph G , let $n_1(G)$ be the number of vertices in G that have degree at least one. We prove the following result.

Theorem 11.2. Let T_1, \dots, T_n be a sequence of trees, $|V(T_i)| = i$ such that for each i , there exists $v_i \in V(T_1)$ such that $n_1(T_i - v_i) \leq \sqrt{(6i)}/4$. Then T_1, \dots, T_n can be packed into K^n .

Our notation is standard, see for example [12]. In particular, by $\lfloor x \rfloor$, the *floor* of x , we mean the greatest integer less than or equal to x , and by $\lceil x \rceil$, the *ceiling* of x , the least integer greater than or equal to x . Define a *spider* to be a tree that contains exactly one vertex of degree greater than 2, and a k -*star* to be a spider of order $k + 1$ that contains a vertex of degree k . Let T be a spider with $u \in V(T)$ such that $d_T(u) > 2$. Denote the components of $T - \{u\}$ by C_1, C_2, \dots, C_m . Define a *leg* of T to be the path $T[V(C_i) \cup \{u\}]$, $1 \leq i \leq m$. We define the *length* of a leg to be the length of the path $T[V(C_i) \cup \{u\}]$. Let L be a leg of T , and $V \in V(L)$ such that $d_L(V) = 1$ and $V \neq u$. We say that V is a *foot* of L . The *distance* between two vertices x, y in a graph G , denoted by $\text{dist}_G(x, y)$ is the length of the shortest path connecting x to y . The *diameter* of a tree T , denoted $\text{diam}(T)$, is the length of the longest path in G . The *neighbors* of a vertex $x \in V(G)$, $\Gamma_G(x)$ is the set of all vertices adjacent in G to x . By $G[X]$, $X \subseteq V(G)$, we mean the subgraph of G induced by X . A *transitive tournament* is a directed graph on n vertices with out degree sequence $(0, 1, \dots, n - 1)$.

CHAPTER 8

GRAPHS WITH GIRTH AT LEAST 7

Our approach to the Erdős-Sós conjecture is motivated by the following elementary exercise. If G is a graph and $\delta(G) \geq q = \lceil k/2 \rceil$, then G contains every tree of order $q + 1$. In solving this exercise, the only obstacle to constructing trees of maximal degree q of larger order is the possibility of triangles in the graph. If one does not allow triangles but does allow 4-cycles, it is not difficult to see that if $q \geq 3$, then the graph $K_{q,q}$ does not contain every tree of order $q + 3$ with maximal degree q . As we are interested in constructing trees of order at least $k + 1$, we will only consider graphs with girth $g \geq 5$.

Perhaps the most obvious question if one wishes to construct a tree T in a graph G with $\delta(G) \geq \lceil k/2 \rceil$ and girth $g \geq 5$, is if one has constructed a subtree T_1 of T of order r , $\Delta(T_1) = \Delta(T)$, and v is a vertex of T_1 of degree 1, how many vertices of T_1 can v be adjacent in G to? Of course, this is a local property of $G[V(T_1)]$, so to investigate this question, it suffices to consider only $G[V(T_1)]$, and then we may take G to be any graph with girth g that contains $G[V(T_1)]$ as an induced subgraph. We may then take T to be any tree that contains T_1 as a subgraph as long as $\Delta(T) = \Delta(T_1)$, and if $x \in V(T_1)$ such that $d_{T_1}(x) \neq 1$, then $d_{T_1}(x) = d_T(x)$. If $g = 5$, then it is easily seen that if $G[V(T_1)]$ is the graph in Figure 1, and $T_1 = G - \{e, e'\}$, then v is adjacent in G to $(r - 1)/2$ vertices of T_1 .

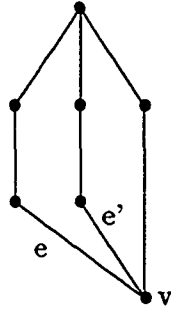


Figure 1. A graph of girth 5 where v is adjacent to $(r - 1)/2$ vertices of T_1 .

If $g = 6$, $G[V(T_1)]$ is the graph in Figure 2, $T_1 = G[V(T_1)] - \{e, e'\}$, then v is adjacent in G to $(r - 2)/2$ vertices of T_1 .

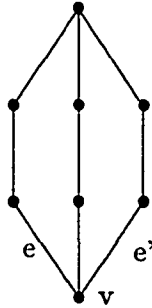


Figure 2. A graph of girth 6 where v is adjacent to $(r - 2)/2$ vertices of T_1 .

The above bounds do not seem to be useful if one wishes to construct trees of order $k + 1$. If $g = 7$, we are able to prove the following useful result.

Theorem 8.1. Let T be a tree of order r such that $\Delta(T) = m$, $\text{diam}(T) \geq t + 1$, $u \in V(T)$ such that $d_T(u) = m$, and v a vertex of T such that $d_T(v) = 1$ and $\text{dist}_T(u, v) \geq t$. If T is contained in a graph G with girth $g \geq 2t + 1$, $t \geq 3$, then v is adjacent in G to most $(r - m - 1)/(t - 1)$ vertices of T .

Proof. Let T be a tree, $u \in V(T)$ such that $d_T(u) = \Delta(T)$. Let $S = \{s \in V(T) : sv \in E(G)\}$, $B_s = \{w \in V(T) : \text{dist}_T(w, s) < t - 1\}$, and $\mathcal{B} = \{B_s : s \in S\}$. Note that the elements of \mathcal{B} are mutually disjoint and that if $B_s \in \mathcal{B}$ then B_s contains exactly one element of S and at least $t - 2$ elements not in S . Hence $|B_s| \geq t - 1$. Further, v is adjacent in T to, say, $w \in V(T)$ and so $|B_w| \geq t$. Let $U = \{x \in V(T) : \text{dist}_T(x, u) \leq 1\}$ and $V = \{s \in S : B_s \cap U \neq \emptyset\}$.

If $V = \emptyset$, then $(t - 1)|S| + m + 1 \leq r$ so

$$|S| \leq \frac{r - m - 1}{t - 1}. \quad (1)$$

Hence we assume that $|V| \neq \emptyset$. As G has girth g , we must have $|V| = 1$.

If $t = 3$ and $u \in S$, then $w \neq u$ and thus there are at least two vertices $x, y \in V(T)$ contained on the unique uv -path in T such that $x, y \notin \cup \mathcal{B}$. Hence

$$\begin{aligned} r &\geq 2|S - 1| + |U| + |V(T) - \cup \mathcal{B}| \\ &\geq 2|S| + m + 1 \end{aligned}$$

and $|S| < (r - m - 1)/(t - 1)$. If $u \notin S$ and $V \neq \emptyset$, then there exists $s \in S$ such that $B_s \cap U \neq \emptyset$. If $s = w$, then there are two vertices of B_w not contained in U . Hence

$$\begin{aligned} r &\geq 2(|S| - 1) + |U| + |B_w - U| \\ &\geq 2(|S| - 1) + m + 1 + 2 \end{aligned} \quad (2)$$

and so (1) follows. If $s \neq w$, then there is at least one vertex of the unique uv -path in T that is not contained in $U \cup (\cup B)$ or there is at least one vertex of B_s not contained in U . Then (2) and (1) follow.

If $t \geq 4$, let $s \in V(T)$ such that $B_s \cap U \neq \emptyset$. If $|S| = 1$, then as $\text{diam}(T) \geq t+1$, the result is trivial. If $|S| \geq 2$ and $|U \cap B_s| = 1$, then at least $t-2$ elements of B_s are not contained in U . If $s \neq w$, then $|B_s - U| \geq t-2$ and $|B_w| \geq t$. Thus

$$r \geq (t-1)(|S|-1) + |U| + |B_s - U| + |B_w| - (t-1)$$

and (1) follows. If $s = w$, then $|B_s - U| \geq t-1$ and

$$r \geq (t-1)(|S|-1) + |U| + |B_s - U|$$

and again (1) follows. If $|U \cap B_s| = 2$, then at least $t-3$ elements of B_s are not contained in U . Let $y \in S$ such that $B_y \cap U = \emptyset$. As G has girth g , there exists a vertex $z \in V(T)$ such that z is contained on the unique yu path in T and $z \notin (\cup B) \cup U$. Then

$$\begin{aligned} r &\geq (t-1)(|S|-1) + |U| + |B_s - U| + |B_w| - (t-1) + |V(T) - (\cup B) \cup U| \\ &\geq (t-1)(|S|-1) + m + 1 + t - 3 + 1 + 1 \\ &= (t-1)|S| + m + 1 \end{aligned} \tag{3}$$

and so (1) follows. Finally, if $|B_s \cap U| > 2$ then $U \subseteq B_s$ and at least $t-4$ vertices of B_s are not contained in U and if $y \in S$ such that $B_y \cap U = \emptyset$, then there are at

least two vertices contained in the unique yu path in T that are not contained in $\cup \mathcal{B}$. Thus (3) and (1) follow. \square

We remark that the bound $(r-m-1)/(t-1)$ given above is best possible. Let T be a spider with $\Delta(T) = m$, such that every leg of T has length t or 1, with at least one leg of length t . Let v be a foot of a leg of length t , and $G[V(T)]$ be the graph with $E(G[V(T)]) = E(T) \cup \{vv' : v' \text{ is a foot of a leg of length } t\}$. Then v is adjacent in G to exactly $(r-m-1)/(t-1)$ vertices of G .

The above result naturally leads to the following bounds for the degree of a vertex v in T given that v is a vertex of degree 1 in some subtree T_1 such that $\Delta(T_1) = \Delta(T)$.

Lemma 8.2. Let T be a tree with maximal degree $m = \Delta(T) \leq \lceil k/2 \rceil$, and T_1 a subtree of T such that $\Delta(T_1) = m$, $|V(T_1)| = r$, and if $y \in V(T_1)$ such that $d_{T_1}(y) > 1$, then $\Gamma_T(y) \subseteq V(T_1)$. Assume $|V(T)| \leq \lfloor \ell k \rfloor + 1$, where

$$\ell \leq \frac{t}{2} - \frac{(t-2)(m-1)}{k}.$$

If $v \in V(T_1)$ such that $d_{T_1}(v) = 1$, then

$$d_T(v) \leq \left\lceil \frac{k}{2} \right\rceil - \max\left(\left\lfloor \frac{r-m}{t} \right\rfloor, 1\right) + 1.$$

Further, let T is a tree of order $k+1$ with degree sequence $(d_1, d_2, \dots, d_{k+1})$, $d_i \geq d_{i+1}$, $d_1 > \lceil k/2 \rceil$, and T_1 is a subtree of T such that $\Delta(T_1) = d_1$, $|V(T_1)| = r$, and

if $y \in V(T_1)$ such that $d_{T_1}(y) > 1$, then $\Gamma_T(y) \subseteq V(T_1)$. If $v \in V(T_1)$ such that $d_{T_1}(v) = 1$, then

$$d_T(v) \leq \left\lceil \frac{k}{2} \right\rceil - \max\left(\left\lfloor \frac{r-m}{2} \right\rfloor, 1\right) + 1.$$

Proof. Let T be a tree with maximal degree $m = \Delta(T) \leq \lceil k/2 \rceil$, and T_1 a subtree as above. Let $v \in V(T_1)$ such that $d_{T_1}(v) = 1$. If $\lfloor (r-m)/t \rfloor = 0$, the result is trivial, so assume $\lfloor (r-m)/t \rfloor \geq 1$. Now $d_T(v) \leq \min(\lfloor \ell k \rfloor - r + 1, m - 1) + 1$, so it suffices to show

$$\left\lceil \frac{k}{2} \right\rceil - \left\lfloor \frac{r-m}{t} \right\rfloor \geq \min(\lfloor \ell k \rfloor - r + 1, m - 1),$$

Note that b is an integer then $\lfloor a \rfloor \geq b$ if and only if $a \geq b$ and $b > \lfloor a \rfloor$ if and only if $b > a$. If $\lfloor \ell k \rfloor - r + 1 \geq m - 1$ then

$$\frac{k}{2} - \frac{r-m}{t} \geq \frac{\ell k}{t} + \frac{(t-2)(m-1)}{t} - \frac{r-m}{t} \geq m-1.$$

Thus

$$\left\lceil \frac{k}{2} \right\rceil - \left\lfloor \frac{r-m}{t} \right\rfloor \geq m-1.$$

If $m-1 > \lfloor \ell k \rfloor - r + 1$ then

$$\frac{(t-2)(\ell k - r + 1)}{t} + \frac{\ell k}{t} - \frac{r-m}{t} > \ell k - r + 1,$$

and

$$\begin{aligned}
 \frac{(t-2)(\ell k - r + 1)}{t} + \frac{\ell k}{t} &\leq \frac{(t-2)(m-1)}{t} + \frac{\ell k}{t} \\
 &\leq \frac{(t-2)(m-1)}{t} + \frac{k}{2} - \frac{(t-2)(m-1)}{t} \\
 &= \frac{k}{2}
 \end{aligned}$$

as $m-1 \geq \ell k - r + 1$. Hence

$$\left\lceil \frac{k}{2} \right\rceil - \left\lfloor \frac{r-m}{t} \right\rfloor > \ell k - r + 1.$$

Let T be a tree of order $k+1$, with T_1 and v as above. Again, $d_T(v) \leq \min(k-r+1, m-1) + 1$, and as $m-1 \geq \lceil k/2 \rceil$, $r \geq \lceil k/2 \rceil + 1$, $m-1 \geq k-r+1$.

Thus it suffices to show

$$\left\lceil \frac{k}{2} \right\rceil - \left\lfloor \frac{r-m}{2} \right\rfloor \geq \min(k-r+1, m-1) = k-r+1.$$

As $m-1 \geq k-r+1$, $k-r+m \geq 2(k-r+1)$ and

$$\left\lceil \frac{k}{2} \right\rceil - \left\lfloor \frac{r-m}{t} \right\rfloor \geq \left\lceil \frac{k}{2} \right\rceil - \left\lfloor \frac{r-m}{2} \right\rfloor \geq k-r+1.$$

□

In the remainder of our work on the Erdős-Sós conjecture we will use this lemma extensively. Essentially, the lemma states that if we wish to construct a tree

T of order $\lfloor \ell k \rfloor + 1$ is a graph G and have constructed a subtree T_1 that satisfies the hypothesis, then to continue the construction inductively we need only show that if $v \in V(T_1)$, $d_{T_1}(v) = 1$, then v is adjacent in G to at most $\lfloor (r - m)/t \rfloor$ vertices of T_1 . Usually, we assume that $\delta(G) \geq \lceil k/2 \rceil$ and that the girth of G is at least $2t + 1$. Combining Theorem 8.1 and Lemma 8.2, we have the following result.

Theorem 8.3. Let G be a graph with girth $g \geq 2t + 1$, $t \geq 3$, and $k \geq 5$. If $\delta(G) \geq \lceil k/2 \rceil$ then G contains every tree T such that $m = \Delta(T) \leq \lceil k/2 \rceil$ of order $\lfloor \ell k \rfloor + 1$, where

$$\ell \leq \frac{t-1}{2} - \frac{(t-3)(m-1)}{k}.$$

Further, if $\Delta(G) \geq d_1 > \lceil k/2 \rceil$, then G contains every tree of order $k + 1$ with degree sequence $(d_1, d_2, \dots, d_{k+1})$, $d_i \geq d_{i+1}$.

Proof. We remark that if $\text{diam}(T) < 2t$, the result is trivial, so we assume without loss of generality that $\text{diam}(T) \geq 2t$. Let $u \in V(T)$ such that $d_T(u) = \Delta(T)$. If $m \leq \lceil k/2 \rceil$, then clearly the subtree of T consisting of u and its neighbors in T is contained in G . If $\Delta(G) \geq d_1 \geq \lceil k/2 \rceil$ and $m \leq d_1$ then the same subtree of T is contained in G . Inductively assume that some subtree T_1 of T has been constructed so that if $v \in V(T_1)$ such that $d_{T_1}(v) > 1$, then $d_{T_1}(v) = d_T(v)$. Let $v \in V(T_1)$ such that $\text{dist}_{T_1}(u, v)$ is minimal and $d_{T_1}(v) = 1$. By construction, the hypothesis of Theorem 8.1 holds, so that v is adjacent to one vertex of T_1 or at most $\lfloor (r - m)/(t - 1) \rfloor$ vertices of T_1 , where $r = |V(T_1)|$. If v is adjacent in G to one vertex of T_1 , then clearly the subtree $T_2 = T[V(T_1) \cup \Gamma_T(v)]$ is contained in G .

If v is adjacent in G to more than one vertex of T_1 , then v is adjacent in G to at most $\lfloor (r-m)/(t-1) \rfloor$ vertices of T_1 , in which case T_2 is contained in G by Lemma 8.2. \square

Corollary 8.4. Let G be a graph with girth $g \geq 7$ and $k \geq 5$. If $|E(G)| \geq \lfloor (k-1)n/2 \rfloor + 1$, then G contains every tree of order $k+1$.

Proof. If $|E(G)| \geq \lfloor (k-1)n/2 \rfloor + 1$, there exists $H \subseteq V(G)$ such that $|E(G[H])| \geq \lfloor (k-1)|H|/2 \rfloor + 1$ and $\delta(G[H]) \geq \lceil k/2 \rceil$. As $|E(G[H])| \geq \lfloor (k-1)|H|/2 \rfloor + 1$, $G[H]$ contains a vertex of degree at least k . Hence the result follows from Theorem 8.3. \square

Corollary 8.5. If G is a graph with girth $g \geq 7$ and $|E(G)| = \lfloor (k-1)n/2 \rfloor$, then if k is odd G does not contain every tree of order $k+1$ if and only if G is $(k-1)$ -regular, and if k is even, then G does not contain every tree of order $k+1$ if and only if at most one vertex of G has degree $k-2$ and every other vertex has degree $k-1$.

Proof. Let G be a graph with girth $g \geq 7$ and $|E(G)| = \lfloor (k-1)n/2 \rfloor$. If there exists $v \in V(G)$ such that $d(v) < (k-1)/2$, then $|E(G-v)| \geq \lfloor (k-1)(n-1)/2 \rfloor + 1$ and so G contains every tree of order $k+1$. If k is odd and there exists $v \in V(G)$ such that $d(v) < k-1$, then there exists $u \in V(G)$ such that $d(u) \geq k$, in which case G contains a k -star. Further, there exists $H \subseteq G$, $H \neq \emptyset$, such that $|E(G[H])| = (k-1)|H|/2$

and $\delta(G[H]) \geq k/2$. Hence G contains every tree T of order $k + 1$ such that $\Delta(T) \leq k/2$. As $G[H]$ contains a vertex of degree $k - 1$, G also contains every tree T of order $k + 1$ such that $k/2 < \Delta(T) \leq k - 1$, and thus G contains every tree of order $k + 1$. If k is even then if there exists $v_1, v_2 \in V(G)$ such that $d(v_1), d(v_2) < k - 1$, then there exists $u \in V(G)$ such that $d(u) \geq k$ and G contains a k -star. By arguments above, G contains every tree of order $k + 1$. \square

CHAPTER 9

GRAPHS WITH GIRTH AT LEAST 5

We now extend Theorem 8.3, and Corollaries 8.4 and 8.5 to graphs with girth at least 5. As the proof of Theorem 9.1 is rather long, it would be useful to preview the main ideas behind the proof. Let G be a graph with girth $g \geq 2t + 1$, $t \geq 2$, and $\delta(G) \geq \lceil k/2 \rceil \geq 3$. Let T be a tree of order $\lfloor \ell k \rfloor + 1$,

$$\ell \leq \frac{t}{2} - \frac{(t-2)(m-1)}{k}$$

where $m = \Delta(T)$. Let T_1 be a subtree of T such that $\Delta(T_1) = \Delta(T)$ and if $d_{T_1}(v) > 1$ then $d_{T_1}(v) = d_T(v)$. Let $r = |V(T_1)|$. By Lemma 8.2, if $v \in V(T_1)$ such that $d_{T_1}(v) = 1$, then if v is adjacent in G to at most $\lfloor (r-m)/t \rfloor$ vertices of T_1 , then the subtree $T_2 = T[V(T_1) \cup \Gamma_T(v)]$ is contained in G . However, we have already observed that if $t = 2$, v can be adjacent in G to at least $(r-1)/2$ vertices of T . Let $w \in V(T_1)$ such that $vw \in E(T_1)$. We can, however, show that there exists $v' \in V(G) - V(T_1)$ such that $wv' \in E(G)$ (we often use Lemma 8.2 to prove this). We would like to show that v' is then adjacent in G to at most $\lfloor (r-m)/t \rfloor$ vertices of T_1 . However, this is also not true, but if it is not true, we show that either $T_2 \subseteq G$ or $d_T(w) = 2$ and there exists $v'' \in V(G) - V(T_1)$, $v'' \neq v'$, such that $wv'' \in E(G)$. Then v'' is adjacent in G to at most $\lfloor (r-m)/t \rfloor$ vertices of T_1 .

To prove that the above occurs, we use counting arguments and Lemma 8.2. The counting arguments can, in general, be complicated. We will, however, give an

example to illustrate the techniques we use in a reasonably simple case. Let $t = 2$, $\lceil k/2 \rceil \geq 4$, and T_1 be the subtree of T in Figure 3. Assume that $vs, vs' \in E(G)$.

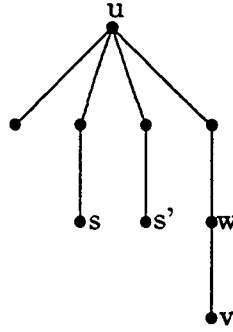


Figure 3. The subtree T_1 .

Then v is adjacent in G to $3 > \lfloor (r - m)/2 \rfloor$ vertices of T_1 . Let $T'_1 = T_1 - \{v\}$, $S'(w) = \{x \in V(T'_1) : wx \in E(G)\}$, $B_x = \{y : \text{dist}_{T_1}(x, y) < t\}$, and $\mathcal{B}'(w) = \{B_x : x \in S'(w)\}$. Assume in our example that $u \notin \cup \mathcal{B}'(w)$. Let C_1, \dots, C_4 be the components of $T'_1 - \{u\}$. Let $s \in C_2$, $s' \in C_3$, and $w \in C_4$. Then, as $g \geq 5$, w is not adjacent in G to any vertex of C_1 , C_2 , or C_3 . Hence w is adjacent to exactly one vertex of T'_1 . Let $r' = |V(T'_1)|$. Then w is adjacent in G to at most $\lfloor (r' - m)/2 \rfloor - 1$ vertices of T'_1 , so by Lemma 8.2 there exists $v' \in V(G) - V(T_1)$ such that $wv' \in E(G)$. Replace v by v' , and let $S(v') = \{x \in V(T_1) : v'x \in E(G)\}$. Let $\mathcal{B}(v') = \{B_x : x \in S(v')\}$, and assume that $u \notin \cup \mathcal{B}(v')$. Let C'_1, \dots, C'_4 be the components of $T_1 - \{u\}$, with $s \in C_2$, $s' \in C_3$, and $v' \in C_4$. Then v' is not adjacent to any vertex of C_1, C_2 , or C_3 , so that $|S(v')| = 1$. Hence v' is adjacent in G to at most $\lfloor (r - m)/2 \rfloor$ vertices of T_1 .

Of course, in general we may not assume that $u \notin \cup \mathcal{B}'(w)$ or that $u \notin \cup \mathcal{B}(v')$, which makes the proof more complicated than simply generalizing the example. The proof, although long, is, as we shall later see, best possible for $k = 6$ if $t = 2$.

Theorem 9.1. Let G be a graph with girth $g \geq 2t + 1$, $t \geq 2$, and $k \geq 5$. If $\delta(G) \geq \lceil k/2 \rceil$ then G contains every tree T such that $m = \Delta(T) \leq \lceil k/2 \rceil$ of order $\lfloor \ell k \rfloor + 1$, where

$$\ell \leq \frac{t}{2} - \frac{(t-2)(m-1)}{k}.$$

Further, if $\Delta(G) \geq d_1 > \lceil k/2 \rceil$, then G contains every tree of order $k + 1$ with degree sequence $(d_1, d_2, \dots, d_{k+1})$, $d_i \geq d_{i+1}$.

Proof. If $g \geq 2t + 1$, $t \geq 2$, observe that a subtree T_1 of T may be constructed inductively in the following fashion:

(i) Select a vertex u of T such that $d_T(u) = \Delta(T) = m$. If $m \leq \lceil k/2 \rceil$, identify u with any vertex of G . If $m > \lceil k/2 \rceil$ and $\Delta(G) \geq d_1 \geq m$, identify u with any vertex of G with degree at least d_1 .

(ii) Assume a subtree T_1 of T has been constructed such that if $y \in V(T_1)$, $d_{T_1}(y) > 1$, then $\Gamma_{T_1}(y) \subseteq V(T_1)$ and if there exists $x \in \Gamma_{T_1}(y)$ such that $d_{T_1}(x) = 1$, then either $d_T(x) = 1$ or if $z \in \Gamma_{T_1}(y)$ such that $\text{dist}_{T_1}(z, u) = \text{dist}_{T_1}(y, u)$, then $d_{T_1}(z) = 1$. Select a vertex $v \in V(T_1)$ such that $d_{T_1}(v) = 1$, $d_T(v) \neq 1$ such that $\text{dist}_{T_1}(u, v)$ is minimal. Such a v always exists unless $T = T_1$. Let $wv \in E(T_1)$ and $N(w) = \{x \in V(T_1) : xw \in E(T_1) \text{ and } \text{dist}_{T_1}(x, u) = \text{dist}_{T_1}(u, v)\}$. If possible,

adjoin each element of $N(w)$ to an appropriate number of vertices of G not contained in $V(T_1)$.

Observe that if (ii) is always possible, then T is contained in G . If $T_b = T[\{v \in V(T) : \text{dist}_T(u, v) \leq t - 1\}]$, then T_b is always constructable, so we may assume that $\text{diam}(T_1) \geq 2t$, and that if v is as in (ii), that $\text{dist}_{T_1}(u, v) \geq t - 1$. By Lemma 8.2, if v is as in (ii) and $r = |V(T_1)|$, then (ii) is possible if every vertex contained in $N(w)$ is adjacent to at most $\lfloor (r - m)/t \rfloor$ vertices of G .

Thus it only remains to show that if T_1 is a subtree of T such that if $y \in V(T_1)$, $d_{T_1}(y) > 1$, then $\Gamma_T(y) \subseteq V(T_1)$, and if there exists $x \in \Gamma_{T_1}(y)$ such that $d_{T_1}(x) = 1$, then either $d_T(x) = 1$ or if $z \in \Gamma_{T_1}(y)$ such that $d_{T_1}(z, u) = \text{dist}_{T_1}(y, u)$, then $d_{T_1}(z) = 1$, then y is adjacent in G to at most $\lfloor (r - m)/t \rfloor$ vertices of T_1 , for every $y \in N(w)$. We will show that although T_1 may not satisfy the above requirements, some subtree of G isomorphic to T_1 does, or that we may continue inductively by some other argument. More specifically, we will show that if $v \in N(w)$ is adjacent in G to more than $\lfloor (r - m)/t \rfloor$ vertices of T_1 , then there exist $v' \in V(G) - V(T_1)$ such that $v'w \in E(G)$ and if $x \in V(G)$ such that $xw \in E(G)$ and $x \notin V(T_1)$ or $x \in V(T_1)$, $x \neq v$, and $d_{T_1}(x) = 1$, then if x is adjacent to more than $\lfloor (r - m)/t \rfloor$ vertices of T_1 , $t = 2$, $d_T(w) = 2$, and if the subtree $T_2 = T[V(T_1) \cup \Gamma_T(v')]$ is not contained in G , then there exists $v'' \in V(G) - V(T_1)$, $v'' \neq v'$, such that v'' is adjacent to at most $\lfloor (r - m)/2 \rfloor$ vertices of T_1 . Then the theorem will follow by induction.

For $x \in V(G)$, let $S(x) = \{s \in V(T_1) : sx \in E(G)\}$, $B_s = \{r \in V(T_1) : \text{dist}_{T_1}(r, s) < t\}$ and $\mathcal{B}(x) = \{B_s : s \in S(x)\}$. Note that the elements of $\mathcal{B}(x)$ are mutually disjoint, and if $s \in S(x)$ then B_s contains exactly one element of $S(x)$ and at least $t - 1$ elements not in $S(x)$. Hence $|B_s| \geq t$. Let $U = \Gamma_{T_1}(u) \cup \{u\}$. We will consider the following cases: Case 1 where there exists $s \in S(v)$ such that $U \subseteq B_s$; Case 2 where $u \notin \cup \mathcal{B}(v)$; and Case 3 where $u \in \cup \mathcal{B}(v)$, but there exists no $s \in S(V)$ such that $U \subseteq B_s$.

Cases 1 and 2 are reasonably straight forward and we will find that if v is adjacent to more than $\lfloor (r - m)/2 \rfloor$ vertices of T_1 , then v' is always adjacent to at most $\lfloor (r - m)/t \rfloor$ vertices of T_1 . Case 3 is more difficult and it is here that v' may be adjacent to more than $\lfloor (r - m)/t \rfloor$ vertices of T_1 , if $t = 2$. It is also in Case 3 that we will have to deal with the situation that generates a best possible result for $k = 6$ and $t = 2$. In general, for all three cases, we usually do not encounter difficulties if $t \geq 3$, and hence if $t \geq 3$, there should be room for improvement.

Case 1. We first consider the case where $u \in S(v)$. If $t = 2$, then $B_u = U$, $u \neq w$, and $|B_w| \geq 3$. Then

$$\begin{aligned} r &\geq 2(|S(v)| - 1) + |U| + |B_w| - 2 \\ &\geq 2|S(v)| - m \end{aligned}$$

and $|S(v)| \leq (r - m)/2$ so $|S(v)| \leq \lfloor (r - m)/2 \rfloor$. If $t \geq 3$, then B_u contains at least $t - 2$ elements of $V(T)$ not in U , $w \neq u$, and $|B_w| \geq t + 1$. Then

$$\begin{aligned}
r &\geq t(|S(v)| - 1) + |U| + |B_w| - 2 \\
&\geq t|S(v)| + m
\end{aligned}$$

and so $|S(v)| \leq \lfloor (r - m)/t \rfloor$.

If $u \notin S(V)$ but $U \subseteq B_s$, then $t \geq 3$. Further, $\text{dist}_{T_1}(u, s) \leq t - 2$ so that $s \neq w$. Note that $|B_s| \geq m + 1 + t - 3$ and again $|B_w| \geq t + 1$. If $|B_w| \geq t + 2$, then

$$\begin{aligned}
r &\geq t(|S(v)| - 1) + |U| + |B_s - U| + |B_w| - t \\
&\geq t(|S(v)| - 1) + m
\end{aligned}$$

and $|S(v)| \leq \lfloor (r - m)/t \rfloor$. Thus we assume $|B_w| \leq t + 1$ and hence that $|B_w| = t + 1$.

We conclude that $d_{T_1}(w) = d_T(w) = 2$. If there exists $x \in V(T_1)$ such that $x \notin \cup \mathcal{B}(v)$, then

$$\begin{aligned}
r &\geq t(|S(v)| - 1) + |U| + |B_s - U| + |B_w| + |V(T_1) - \cup \mathcal{B}(v)| \\
&\geq t|S(v)| + m
\end{aligned}$$

and again $|S(v)| \leq \lfloor (r - m)/t \rfloor$. Thus we assume $\cup \mathcal{B}(v) = V(T_1)$. Suppose $|S(w)| > 2$. Then there exists $x \in S(w)$ such that $x \notin B_w$. As $x \in \mathcal{B}(v)$, there exists $s' \in S(v)$ such that $x \in B_{s'}$, so that $\text{dist}_{T_1}(x, s') \leq t - 1$. Let P be the unique xs' -path in T_1 . Then $wPvw$ is a cycle of length at most $t + 2$, contradicting the hypothesis that $t \geq 3$. Hence $|S(w)| = 2$. As $\lceil k/2 \rceil \geq 3$, there exists $v' \in V(G) - V(T_1)$ such that $v'w \in E(G)$.

We will show that $|S(v')| = 1$. If $|S(v')| > 1$, then there exists $x \in S(v')$, $x \neq w$, such that $xv' \in E(G)$. As $x \in \cup \mathcal{B}(v)$, there exists $s' \in S(v)$ such that

$x \in B_{s'}$ and $\text{dist}_{T_1}(x, s') \leq t-1$. Let P be the unique xs' -path in T_1 . Then $vwv'Pv$ is a cycle in G of length at most $t+3 \geq 2t+1$ so that $t \leq 2$, a contradiction. Hence $|S(v')| = 1$. This completes the argument for Case 1.

Let $T'_1 = T_1 - N(w)$, $r' = |V(T'_1)|$, $S'(w) = \{x \in V(T'_1) : xw \in E(G)\}$, and $B'(w) = \{B_{s'} : s' \in S'(w)\}$. Before proceeding to Case 2, we will show that if v is adjacent in G to more than $\lfloor (r-m)/t \rfloor$ vertices of T_1 and $T_2 \not\subseteq G$, then either there exist $v' \in V(G) - V(T_1)$ such that $wv' \in E(G)$ or that $d_T(w) \geq 3$.

If there exists no $v' \in V(G) - V(T_1)$ such that $wv' \in E(G)$, then w is adjacent in G to at least $\lceil k/2 \rceil$ vertices of T_1 . Assume $d_T(w) = 2$. Then $|B_w| \geq t+1$, and if $y \in S(v)$, then $y \notin \cup B'(w)$. Hence

$$\lfloor \ell k \rfloor \geq r \geq t \left(\left\lceil \frac{k}{2} \right\rceil - 2 \right) + t + 1 + |S(v)| - 1$$

and

$$\frac{tk}{2} - (t-2)(m-1) \geq \frac{tk}{2} - t + |S(v)|.$$

Thus

$$0 \leq (t-2)(m-1) \leq t - |S(v)|$$

so that $|S(v)| \leq t$. As $|S(v)| \leq t$, either $T_2 \subseteq G$ or $d_T(v) \geq \lceil k/2 \rceil - t + 2$. Hence

$$r \leq \lfloor \ell k \rfloor + 1 - \left(\left\lceil \frac{k}{2} \right\rceil - t + 1 \right)$$

and

$$\frac{tk}{2} - (m-1)(t-2) - \left\lceil \frac{k}{2} \right\rceil + t \geq \frac{tk}{2} - t + |S(v)|.$$

Thus

$$\left\lceil \frac{k}{2} \right\rceil \leq 2t - (m-1)(t-2) - |S(v)|.$$

If $m = 2$, then either $T_2 \subseteq G$ or v is adjacent in G to at least $\lceil k/2 \rceil$ vertices of T_1 , and hence that $r \geq t\lceil k/2 \rceil$, a contradiction. Hence we assume $m \geq 3$. Note that $|S(v)| \geq 2$ or $T_2 \subseteq G$. Thus

$$\left\lceil \frac{k}{2} \right\rceil \leq 2t - 2(t-2) - 2 = 2,$$

a contradiction.

In the arguments for both Case 2 and Case 3, we first show that if $|S(v)| \geq \lfloor (r-m)/t \rfloor$, then either $T_2 \subseteq G$ or there exists $v' \in V(G) - V(T_1)$ such that $vv' \in E(G)$.

Case 2. We first consider when $U \cap (\cup \mathcal{B}(v)) = \emptyset$. If $U \cap (\cup \mathcal{B}(v)) = \emptyset$, then $t|S(v)| + m + 1 \leq r$ and so $|S(v)| \leq \lfloor (r-m)/t \rfloor$. For the remainder of Case 2 we assume without loss of generality that $U \cap (\cup \mathcal{B}(v)) \neq \emptyset$.

Let C'_1, C'_2, \dots, C'_m be the components of $T_1 - u$. One can easily verify that $\cup \mathcal{B}(v) \subseteq \cup_{i \leq m} V(C'_i)$. Let $Q = \{i : V(C'_i) \subseteq (\cup \mathcal{B}(v)) \text{ and whenever } s \in S(v) \cap V(C'_i), \text{ then } |B_s| = t\}$. Clearly if $i \in Q$, then $|V(C'_i)| = at$, $a \geq 1$. Let $R = \{i : |V(C'_i)| = t\} \cup Q$, and $a = |R|$. If $|V(C'_i)| \leq t - 1$, then $V(C'_i) \cap (\cup \mathcal{B}) = \emptyset$, and if $|V(C'_i)| \geq t + 1$, $i \notin Q$, then either there exists $s \in S(v)$ such that $|B_s| \geq t + 1$ or there exists $y \in V(C'_i)$ such that $y \notin \cup \mathcal{B}(v)$. Let $d(i) = |V(C'_i)| - 1$ if $|V(C'_i)| \leq t - 1$, $d(i) = 0$ if $i \in R$, and

$$d(i) = \sum_{s^* \in S(v) \cap V(C'_i)} (|B_{s^*}| - t) + |V(C'_i) - \cup_{s^* \in S(v) \cap V(C'_i)} B_{s^*}| - 1$$

if $|V(C'_i)| \geq t + 1$ and $i \notin Q$. Let $d = \sum_{i=1}^m d(i) + 1$. We will count the vertices of $V(T_1)$ in the following fashion: from each $s^* \in S$, we will count t vertices, and from each $i \notin R$, we will count one vertex. This will not count every vertex of $V(T_1)$, so we define $d(i)$ to count the vertices of C'_i that have not already been counted, and d to count all of the vertices of T_1 that have not been counted. If $|Q| \leq (a(t-1) + d)/t$, then

$$\begin{aligned} r &\geq t|S(v)| + m - a + d + t(|R - Q|) \\ &\geq t|S(v)| + m - a + d + t\left(a - \frac{a(t-1) + d}{t}\right) \\ &= t|S(v)| + m \end{aligned}$$

and $|S(v)| \leq \lfloor (r - m)/t \rfloor$. Thus we assume that $|Q| > (a(t-1) + d)/t$. By arguments above, we may assume $d_T(w) \geq 3$ so that if $w \in V(C'_f)$, then $d(f) \geq 1$ and $d \geq 2$.

Hence $a \geq 3$.

Assume there exists $i \in Q$ such that $V(C'_i) \cap (\cup \mathcal{B}'(w)) \neq \emptyset$. Then there exists $x \in S(v) \cap V(C'_i)$ and $s' \in \mathcal{B}'(w)$ such that $B_x \cap B_{s'} \neq \emptyset$. Let $y \in B_x \cap B_{s'}$. Let P be the unique xs' -path in T_1 . Then $vPwv$ is a cycle in G , so that $\text{dist}_{T_1}(x, s') \geq 2t - 2$. As $B_x \cap B_{s'} \neq \emptyset$, $\text{dist}_{T_1}(x, s') = 2t - 2$ and $B_x \cap B_{s'} = \{y\}$. Clearly $s' \notin V(C'_i)$ and thus $u \in B_{s'}$. We conclude that $U \subseteq B_{s'}$. Then

$$(\cup_{j \in R} V(C'_j) - U) \cap (\cup \mathcal{B}'(w)) = \emptyset.$$

Hence

$$\begin{aligned} r' &\geq t(|S'(w)| - 1) + |U| + |\cup_{j \in R} V(C'_j) - U| \\ &\geq t(|S'(w)| - 1) + m + 1 + (t - 1)a \\ &= t|S'(w)| + m + (a - 1)(t - 1). \end{aligned}$$

Thus

$$|S'(w)| \leq \frac{r' - m}{t} - \frac{(a - 1)(t - 1)}{t}.$$

If $(a - 1)(t - 1) \geq t$, then $|S'(w)| \leq \lfloor (r' - m)/t \rfloor - 1$. By Lemma 8.2, there exists $v' \in V(G) - V(T_1)$ such that $v'w \in E(G)$. If $(a - 1)(t - 1) < t$ then

$$t < \frac{a - 1}{a - 2}.$$

As $a \geq 3$, $t < 2$, a contradiction. This completes the argument for the case where $V(C'_i) \cap (\cup \mathcal{B}(v)) \neq \emptyset$ for some $i \in Q$.

If $V(C'_i) \cap (\cup \mathcal{B}'(w)) = \emptyset$ for all $i \in Q$, we first consider when $u \notin \cup \mathcal{B}'(w)$. Let $w \in V(C'_f)$. Assume there exists $i \notin R$, $i \neq f$, such that there exists no $s^* \in S'(w) \cap V(C'_i)$ such that $|B_{s^*}| \geq t+1$ and $V(C'_i) \subseteq \cup \mathcal{B}'(w)$. Then $|V(C'_i)| \geq 2t$ and $V(C'_i) \cap (\cup \mathcal{B}(v)) = \emptyset$. Hence $d(i) \geq 2t-1$. Let c be the number of such i . Then

$$\begin{aligned} r' &\geq t|S'(w)| + m - a - c - 1 + 1 + t|Q| \\ &> t|S'(w)| + m + a(t-2) + c(2t-2) + 1. \end{aligned}$$

As $c \geq 1$, $a(t-2) + c(2t-2) + 1 \geq 2t-1$ and $|S'(w)| \leq \lfloor (r-m)/t \rfloor - 1$. Thus we assume without loss of generality that no such i exists. Then

$$\begin{aligned} r' &\geq t|S'(w)| + m - a - 1 + 1 + t|Q| \\ &> t|S'(w)| + m + a(t-2) + d. \end{aligned}$$

Hence

$$|S'(w)| < \frac{r' - m}{t} - \frac{d + a(t-2)}{t}.$$

If $d + a(t-2) \geq t$, then $|S'(w)| \leq \lfloor (r'-m)/t \rfloor - 1$. Hence we assume $d + a(t-2) < t$.

Then

$$t < 2 + \frac{2-d}{a-1}.$$

As $a \geq 3$ and $d \geq 2$, $t < 2$, a contradiction.

If $u \in \cup \mathcal{B}'(w)$, then as $V(C'_i) \cap (\cup \mathcal{B}'(w)) = \emptyset$ for all $i \in Q$, there exists no $s^* \in S'(w)$ such that $U \subseteq B_{s^*}$. Let $1 \leq j \leq m$ such that there exists $s' \in S'(w)$ such

that $u \in B_{s'}$, and $s' \in V(C'_j)$. Let $C_i = T'_1[V(C'_i)]$ if $i \neq j$ and $C_j = T_1[V(C'_j) \cup \{u\}]$. Note that $j \notin Q$, and that if $i \in Q$, then $(\cup B'(w)) \cap V(C_i) = \emptyset$. If $i \notin R$, $j \neq i \neq f$, then with arguments analogous to those above we may assume without loss of generality that either there exists $s^* \in V(C_i) \cap S(w)$ such that $B_{s^*} \subseteq V(C_i)$ and $|B_{s^*}| \geq t + 1$ or there exists $y \in V(C_i)$ such that $y \notin \cup B'(w)$. Hence

$$\begin{aligned} r' &\geq t|S'(w)| + m - a - 2 + t|Q| \\ &> t|S'(w)| + m + a(t - 2) + d - 2. \end{aligned}$$

Thus

$$|S'(w)| < \frac{r' - m}{t} - \frac{d - 2 + a(t - 2)}{t}.$$

If $d - 2 + a(t - 2) \geq t$, then $|S'(w)| \leq \lfloor (r' - m)/t \rfloor - 1$. If $d - 2 + a(t - 2) < t$, then

$$t < 2 + \frac{4 - d}{a - 1}.$$

Assume there does not exist $v' \in V(G) - V(T_1)$ such that $v'w \in E(G)$. Then w is adjacent in G to at least $\lceil k/2 \rceil$ vertices of T_1 . Note that $d \geq d_T(w) - 1$ and $|B_w| \geq d_T(w) + t - 1$. Hence

$$\begin{aligned} \lfloor \ell k \rfloor &\geq r \geq t \left(\left\lceil \frac{k}{2} \right\rceil - d_T(w) \right) + d_T(w) + t - 1 + t|Q| \\ &> \frac{tk}{2} - (t - 2)d_T(w) + t - 2 + (t - 1)a. \end{aligned}$$

Hence $t \geq 3$. Thus we may assume that $a = 3$. Then $|Q| = 3$,

$$\begin{aligned}
r' &\geq t|S'(w)| + m - 3 - 2 + 3t \\
&= t|S'(w)| + m + 3t - 5,
\end{aligned}$$

and $|S'(w)| \leq \lfloor (r' - m)/t \rfloor - 1$.

Let $x \in V(G)$ such that $xw \in E(G)$ and either $xw \notin V(T_1)$ or $x \in V(T_1)$ and $d_{T_1}(x) = 1$, $x \neq v$. We will show that $|S(x)| \leq \lfloor (r - m)/t \rfloor$. We consider the following subcases: Case 2.1 where there exists $s'' \in S(x)$ such that $U \subseteq B_{s''}$; Case 2.2 where $u \notin \cup \mathcal{B}(x)$; and Case 2.3 where there exists no $s'' \in S(x)$ such that $U \subseteq B_{s''}$ but there exists $s'' \in S(x)$ such that $u \in B_{s''}$.

Case 2.1 Suppose there exists $s'' \in S(x)$ such that $U \subseteq B_{s''}$. By arguments in Case 1, $|S(x)| \leq \lfloor (r - m)/t \rfloor$.

Case 2.2 Suppose $U \cap (\cup \mathcal{B}(x)) \neq \emptyset$. Let $i \in Q$ and observe that if $t \geq 3$, then $V(C'_i) \cap (\cup \mathcal{B}(x)) = \emptyset$. Let $t = 2$ and $i \in Q$. If $|V(C'_i)| = 2$, then as $u \notin \cup \mathcal{B}(x)$, $V(C'_i) \cap (\cup \mathcal{B}(x)) = \emptyset$. If $|V(C'_i)| > 2$, then we claim that $|V(C'_i)| \geq 2|V(C'_i) \cap S(x)| + 2$. Let $y \in V(C'_i)$ such that $y \in U$. As $u \notin \cup \mathcal{B}(x)$, $xy \notin E(G)$ but as $i \in Q$, there exists $x_1 \in S(v)$ such that $y \in B_{x_1}$. As $|V(C'_i)| > 2$, there exists $x_2 \in V(C'_i)$ such that $x_2y \in E(T_1)$ or $x_2x_1 \in E(T_1)$. If $x_2x_1 \in E(T_1)$, then $|B_{x_1}| \geq 3$. Hence $x_2y \in E(T_1)$. Now $x_2v \notin E(G)$ and thus there exists $x_3 \in V(C'_i) \cap S(v)$ such that $x_3x_2 \in E(G)$ and $x_2 \in B_{x_3}$. Further, if $d_{T_1}(x_3) > 2$, then $|B_{x_3}| \geq 3$ so that $d_{T_1}(x_3) = 1$. Similarly, $d_{T_1}(x_1) = 1$. Then either the claim is true or $x_2x \in E(G)$. If $x_2x \in E(G)$, then $|B_{x_2}| = 3$ and $x_1 \notin \cup \mathcal{B}(x)$ and the claim follows.

If $|V(C'_i)| \leq t-1$ then $V(C'_i) \cap (\cup \mathcal{B}(x)) = \emptyset$. If $|V(C'_i)| \geq t+1$, $i \notin R$, first assume there exists no $s^* \in S(x)$ such that $|B_{s^*}| \geq t+1$ and $V(C'_i) \subseteq \cup \mathcal{B}(x)$. Clearly $|V(C'_i)| \geq 2t$. Let c be the number of such i . If $t \geq 3$, then v cannot be adjacent to any vertex of C'_i and $d(i) \geq 2t-1$. Then

$$\begin{aligned} r &\geq t|S(x)| + m - a - c + 1 + t|Q| \\ &> t|S(x)| + m + a(t-2) + c(2t-2) + 2 \end{aligned}$$

and $|S(x)| \leq \lfloor (r-m)/t \rfloor$. If $t = 2$, then by arguments analogous to those above, $d(i) \geq 1$. Then

$$\begin{aligned} r &\geq t|S(x)| + m - a - c + 1 + t|Q| \\ &> t|S(x)| + m + a(t-2) + 1 \end{aligned}$$

and $|S(x)| \leq \lfloor (r-m)/2 \rfloor$. If no such i exists, then

$$r > t|S(x)| + m + a(t-2) + d$$

and $|S(x)| \leq \lfloor (r-m)/t \rfloor$.

If $U \cap (\cup \mathcal{B}(x)) = \emptyset$, then $t|S(x)| + m + 1 \leq r$ and so $|S(x)| \leq \lfloor (r-m)/t \rfloor$.

Case 2.3 If there exists no $s'' \in S(x)$ such that $U \subseteq B_{s''}$ but there exists $s'' \in S(x)$ such that $u \in B_{s''}$, then $s'' \neq u$. Let $1 \leq j \leq m$ such that $s'' \in V(C'_j)$. Let $C_i = C'_i$ if $i \neq j$ and $C_j = T_1[V(C'_j) \cup \{u\}]$. If $|V(C_i)| \leq t-1$, then $V(C_i) \cap (\cup \mathcal{B}(x)) = \emptyset$ and if $i \in Q$, then by arguments above if $t \geq 3$ then $V(C'_i) \cap (\cup \mathcal{B}(x)) = \emptyset$. If

$t = 2$, $i \in Q$, and $|V(C'_i)| > 2$, then $|V(C'_i)| \geq 2|V(C'_i) \cap S(x)| + 2$. Let c be as in Case 2.2. Then

$$\begin{aligned} r &\geq t|S(x)| + m - a - 1 - c + t|Q| \\ &> t|S(x)| + m + a(t - 2) - c - 1 + d. \end{aligned}$$

By arguments in Case 2.2, if $t \geq 3$ then $d \geq (2t - 1)c + 1$ and $|S(x)| \leq \lfloor (r - m)/t \rfloor$. If $t = 2$, then $d \geq c + 1$ and again $|S(x)| \leq \lfloor (r - m)/2 \rfloor$. Hence we assume without loss of generality that if $i \notin R$, $i \neq j$, and $|V(C_i)| \geq t + 1$, then there exists $s^* \in S(x) \cap V(C_i)$ such that $|B_{s^*}| \geq t + 1$. If $j \notin Q$, then for every $i \in Q$, $V(C_i) = V(C'_i)$. Hence

$$r > t|S(x)| + m + a(t - 2) + d - 1$$

and, as $d \geq 1$, $|S(x)| \leq \lfloor (r - m)/t \rfloor$.

If $j \in Q$, then $|B_{s''}| = nt + 1$, $n \geq 1$. If $i \in Q$, $i \neq j$, then $V(C_i) = V(C'_i)$ and by arguments above if $t \geq 3$, then $V(C_i) \cap (\cup B(x)) = \emptyset$, and if $t = 2$ and $|V(C_i)| > 2$, then $|V(C_i)| \geq 2|V(C_i) \cap S(x)| + 2$. Then

$$r > t|S(x)| + m + a(t - 2) + 1 + d - t$$

If $t = 2$, then as $d \geq 1$, $|S(x)| \leq \lfloor (r - m)/2 \rfloor$. If $t \geq 3$, then

$$|S(x)| < \frac{r - m}{t} + 1 - \frac{a(t - 2) + 1 + d}{t}.$$

If $a(t-2) + 1 + d \geq t$, then $|S(x)| \leq \lfloor (r-m)/t \rfloor$. If $a(t-2) + 1 + d < t$, then

$$t < 2 + \frac{1-d}{a-1}.$$

As $d \geq 1$, $a \geq 2$ so that $t < 2$, a contradiction.

We now have arrived at Case 3, the most difficult part of the proof.

Case 3. If $u \in \cup \mathcal{B}(v)$, let $s \in S(v)$ such that $u \in B_s$. Let C'_1, C'_2, \dots, C'_m be as in Case 2, and $C_i = C'_i$ if $s \notin V(C'_i)$ and $C_i = T_1[V(C'_i) \cup \{u\}]$ if $s \in V(C'_i)$. Let $Q = \{i : V(C_i) \subseteq (\cup \mathcal{B}(v))\}$ and if $s \in S(v) \cap V(C_i)$, then $|B_s| = t\}$, $R = \{i : |V(C_i)| = t\} \cup Q$, and $a = |R|$. If $|V(C_i)| \leq t-1$ then $V(C_i) \cap (\cup \mathcal{B}(v)) = \emptyset$, and if $|V(C_i)| \geq t+1$, $i \notin Q$, then either there either $s^* \in S(v)$ such that $|B_{s^*}| \geq t+1$ or there exists $y \in V(C_i)$ such that $y \notin \cup \mathcal{B}(v)$. If $R = \emptyset$, then $t|S(v)| + m \leq r$ and so $|S(v)| \leq \lfloor (r-m)/t \rfloor$. Thus we assume $R \neq \emptyset$. Let $d(i) = |V(C_i)| - 1$ if $|V(C_i)| \leq t-1$, $d(i) = 0$ if $i \in R$ and

$$d(i) = \sum_{s^* \in S(v) \cap V(C_i)} (|B_{s^*}| - t) + |V(C_i) - \cup_{s^* \in S(v) \cap V(C_i)} B_{s^*}| - 1$$

if $|V(C_i)| \geq t+1$ and $i \notin R$. Let $d = \sum_{i=1}^m d(i)$.

Assume there exists $s' \in S'(w)$ and $i, j, i \neq j$, such that $B_{s'} \cap V(C_i) \neq \emptyset$ and $B_{s'} \cap V(C_j) \neq \emptyset$. Then we may assume $u \in V(C_i)$ or $u \in V(C_j)$. Without loss of generality assume $u \in V(C_i)$. Let $s' \in V(C_j)$. Then clearly $\text{dist}_{T_1}(s, u) = t-1$ and $\text{dist}_{T_1}(s', u) \leq t-1$. Denote by P_1 the unique su -path in T_1 and let P_2 be the

unique us' -path in T_1 . Then vP_1P_2wv is a cycle in G of length at most $2t + 1$, so that $\text{dist}_{T_1}(s', u) = t - 1$. Hence $\text{dist}_{T_1}(u, w) \geq t + 1$. As $\text{dist}_{T_1}(u, w) \geq t + 1$, if $w \in V(C_f)$ then $d(f) \geq 2$ so that $d \geq 2$, or there exists $x \in V(C_f)$, $x \neq w$, such that $xv \in E(G)$. Then $x \notin \cup \mathcal{B}'(w)$. Let α be the number of such x . Note that $d + \alpha \geq 2$. If $|Q| \leq (a(t - 1) + d)/t$ then

$$r \leq t|S(v)| + m - a + t \left(a - \frac{a(t - 1) + d}{t} \right) + d$$

and $|S| \leq \lfloor (r - m)/t \rfloor$. Thus we assume $|Q| > (a(t - 1) + d)/t$. If $j \in Q$, then clearly G contains a cycle of size at most $t + 2$, so that $j \notin Q$. Let $C_h^* = C_h[V(T'_1) \cap V(C_h)]$. Then $C_h^* = C_h$ unless $h = f$. Let $D_h = C_h^*$ if $h \neq i, j$, $D_j = T'_1[\{u\} \cup V(C_j)]$, and $D_i = C_i - \{u\}$. Let $Z = \{h \notin R : V(D_h) \subseteq \cup \mathcal{B}'(w) \text{ and there exists no } s^* \in S'(w) \cap V(D_h) \text{ such that } |B_{s^*}| \geq t + 1\}$, and $c = |Z|$. Note that if $h \notin R \cup Z$, then there exist $s^* \in V(D_h) \cap S'(w)$ such that $|B_{s^*}| \geq t + 1$ or there exists $y \in V(C_h)$ such that $y \notin \cup \mathcal{B}'$. Further, if $h \in Z$, then $d(h) \geq 2t - 1$ so that $d \geq c(2t - 1) + 1$.

If $i \in Q$ and $j \in R - Q$ then $|V(D_i)| = nt - 1$, $n \geq 1$, and $|V(D_j)| = t + 1$. Then there are at least $m - a - c + 2$ D_h such that either there exists $y \in V(D_h)$ such that $y \notin \cup \mathcal{B}'(w)$ or there exists $s^* \in S'(w) \cap V(D_h)$ such that $|B_{s^*}| \geq t + 1$. Hence

$$r' \geq t|S'(w)| + m - a - c + 2 + \alpha + t(|Q| - 1)$$

and

$$|S'(w)| < \frac{r' - m}{t} - \frac{ta - 2a + d + 2 - t - c + \alpha}{t}.$$

If $c \geq 1$, then $|S'(w)| \leq \lfloor (r' - m)/t \rfloor - 1$. Hence we assume without loss of generality that $c = 0$. As $j \in R - Q$ and $d \geq 1$, $a \geq 3$. As $t \geq 2$ and $\alpha + d \geq 2$, $ta - 2a + d + 2 - t + \alpha \geq t$, and $|S'(w)| \leq \lfloor (r' - m)/t \rfloor - 1$.

If $i \notin R$ and $j \in R - Q$, then $|V(C_i)| \geq t + 1$ and $\text{diam}(C_i) \geq t$. Thus $|V(D_i)| \geq t$. As $|V(D_j)| = t + 1$

$$\begin{aligned} r' &\geq t|S'(w)| + m - a - c + \alpha + t|Q| \\ &> t|S'(w)| + m + a(t - 2) - c + d + \alpha. \end{aligned}$$

Hence

$$|S'(w)| < \frac{r' - m}{t} - \frac{a(t - 2) - c + d + \alpha}{t}.$$

If $c \geq 1$, then $|S'(w)| \leq \lfloor (r' - m)/t \rfloor - 1$. If $c = 0$, then as $d \geq 1$, $a \geq 2$, and as $t \geq 2$ and $\alpha + d \geq 2$, $a(t - 2) - d - \alpha \geq t$. Hence $|S'(w)| \leq \lfloor (r' - m)/t \rfloor - 1$.

If $i \in Q$ and $j \notin R$ then $|V(D_i)| = t - 1$. If $|V(C_j)| \geq t + 1$, then $|V(D_j)| \geq t + 2$.

Further, $V(D_i) \cap (\cup B'(w)) = \emptyset$. Hence

$$r' \geq t|S'(w)| + m - a - c - 1 + t - 1 + \alpha + t(|Q| - 1)$$

and

$$|S'(w)| \leq \frac{r' - m}{t} - \frac{ta - 2a - 2 - c + d + \alpha}{t}.$$

If $c \geq 1$, then $|S'(w)| \leq \lfloor (r' - m)/t \rfloor - 1$. Hence we assume $c = 0$. As $c = 0$, either there exists $s^* \in S'(w) \cap V(D_j)$ such that $|B_{s^*}| \geq t + 1$ or there exists $y \in V(D_j)$ such that $y \notin \cup \mathcal{B}'(w)$. Hence

$$r' \geq t|S'(w)| + m - a + t - 1 + \alpha + t(|Q| - 1)$$

and

$$|S'(w)| \leq \frac{r' - m}{t} - \frac{ta - 2a + d + \alpha}{t}.$$

As $d \geq 1$, $a \geq 2$. As $t \geq 2$ and $\alpha + d \geq 2$, $ta - 2a - 1 + d \geq t$. Hence $|S'(w)| \leq \lfloor (r' - m)/t \rfloor - 1$.

If $|V(C_j)| \leq t - 1$, then $|V(C_j)| = t - 1$ and $d(j) = t - 2$. Hence $d \geq t - 1$, and $\alpha + d \geq t$. As before, $V(D_i) \cap (\cup \mathcal{B}'(w)) = \emptyset$. Hence

$$\begin{aligned} r' &\geq t|S'(w)| + m - a - 1 + t - 1 + \alpha + t(|Q| - 1) \\ &> t|S'(w)| + m + a(t - 2) - 2 + d + \alpha. \end{aligned}$$

Thus

$$|S'(w)| \leq \frac{r' - m}{t} - \frac{a(t - 2) - 1 + d + \alpha}{t}$$

Hence we may assume $t = 2$. If $\alpha + d \geq 3$, then $|S'(w)| \leq \lfloor (r' - m)/2 \rfloor - 1$. Hence we may also assume that $\alpha + d \leq 2$.

If there does not exist $v' \in V(G) - V(T_1)$ such that $wv' \in E(G)$, then w is adjacent in G to at least $\lceil k/2 \rceil$ vertices of T_1 . Then $\alpha + d = 2$ so that $d_T(w) = 3$. Hence $|B_w| = 4$. As $d \geq 1$, $|Q| \geq 2$. Thus there are at least 3 vertices of $\cup_{h \in Q} V(C_h)$ that are not contained in $\cup \mathcal{B}'(w)$. Hence

$$r \geq 2(\lceil k/2 \rceil - 2) + 4 + 3 \geq k + 1,$$

a contradiction.

If $i \notin R$ and $j \notin R$, then $|V(C_i)| \geq t + 1$ and $|V(C_j)| \neq t$. If $|V(C_j)| \geq t + 1$ then as $\text{dist}_{T_1}(s', u) = t - 1$, $|B_{s'}| \geq t + 1$. If v is adjacent in G to some element of $V(C_j)$ then there exists $y \in V(D_j)$ such that $y \notin \cup \mathcal{B}'(w)$. Hence

$$\begin{aligned} r' &\geq t|S'(w)| + m - a - c + \alpha + t|Q| \\ &> t|S'(w)| + m + a(t - 2) - c + d + \alpha. \end{aligned}$$

If $c > 0$, then $d \geq c(2t - 1) + 1$ and $|S'(w)| \leq \lfloor (r' - m)/t \rfloor - 1$. If $c = 0$, then

$$|S'(w)| < \frac{r' - m}{t} - \frac{a(t - 2) + \alpha + d}{t}.$$

As $t \geq 2$ and $d + \alpha \geq 2$, $|S'(w)| \leq \lfloor (r' - m)/t \rfloor - 1$.

If v is not adjacent in G to some element of $V(C_j)$, then $d(j) \geq t$ and

$$r' \geq t|S'(w)| + m - a - c - 1 + \alpha + t|Q|.$$

If $c > 0$ then again $|S'(w)| \leq \lfloor (r' - m)/t \rfloor - 1$. If $c = 0$, then

$$|S'(w)| < \frac{r' - m}{t} - \frac{a(t - 2) + d - 1 + \alpha}{t}.$$

As $d(j) \geq t$, $d \geq t + 1$ and $|S'(w)| \leq \lfloor (r' - m)/t \rfloor - 1$.

If $|V(C_j)| \leq t - 1$, then $|V(C_j)| = t - 1$. Thus $|V(D_j)| = t$. If $|V(C_i)| = t + 1$, then $V(D_i) \cap (\cup B'(w)) = \emptyset$ and so

$$r' \geq t|S'(w)| + m - a - c - 2 + t + \alpha + t|Q|$$

If $c > 0$, then as usual $|S'(w)| \leq \lfloor (r' - m)/t \rfloor - 1$. If $c = 0$, then

$$|S'(w)| < \frac{r' - m}{t} - 1 - \frac{ta - 2a + \alpha + d - 2}{t}.$$

As $t \geq 2$ and $d + \alpha \geq 2$, $|S'(w)| \leq \lfloor (r' - m)/t \rfloor - 1$. This completes the argument for there existing and $i \neq j$ such that $B_{s'} \cup V(C_i) \neq \emptyset$.

If there exists no $i \neq j$ such that $B_{s'} \cap V(C_i) \neq \emptyset$ and $B_{s'} \cap V(C_j) \neq \emptyset$, then $u \notin \cup B'(w)$. Then we may again assume $|Q| > (a(t - 1) + d)/t$. Further,

$$r' \geq t|S'(w)| + m - a - c + 1 + t|Q|$$

If $c > 0$, then $|S'(w)| \leq \lfloor (r' - m)t \rfloor - 1$. If $c = 0$, then

$$|S'(w)| \leq \frac{r' - m}{t} - \frac{ta - 2a + d + 1}{t}.$$

As $d \geq 1$ and $t \geq 2$, $|S'(w)| \leq \lfloor (r' - m)/t \rfloor - 1$.

We now show that if $x \in V(G) - V(T_1')$ such that $x \neq v$ and $wx \in E(G)$, then if x is not adjacent in G to at most $\lfloor (r - m)/t \rfloor$ vertices of T_1 , then $t = 2$, $d_T(w) = 2$, and if the subtree T_2 is not contained in G , then there exists $v'' \in V(G) - V(T_1)$, $v'' \neq v'$, such that $wv'' \in E(G)$ and v'' is adjacent to at most $\lfloor (r - m)/2 \rfloor$ vertices of T_1 .

Let x be as immediately above. We consider the following subcases: Case 3.1 where there exists $s'' \in S(x)$ such that $U \subseteq B_{s''}$; Case 3.2 where $u \notin \cup B(x)$; and Case 3.3 where there exists no $s'' \in S(x)$ such that $U \subseteq B_{s''}$ but there exists $s'' \in S(x)$ such that $u \in B_{s''}$.

Case 3.1. Suppose there exists $s'' \in S''$ such that $U \subseteq B_{s''}$. Then this case follows with arguments analogous to those in Case 1.

Case 3.2. By Case 2, if $|S(x)| > \lfloor (r - m)/t \rfloor$, then $|S(v)| \leq \lfloor (r - m)/t \rfloor$, a contradiction.

The next subcase, Case 3.3, is where we actually encounter the situation that will be best possible for $k = 6$ and $t = 2$.

Case 3.3. Suppose $u \in \cup B(x)$, but $U \not\subseteq \cup B(x)$. Let $s'' \in S(x)$ such that $u \in B_{s''}$, and let $s'' \in V(C_i)$. First assume $u \in V(C_i)$. Then $|V(C_i)| \geq t + 1$.

If $j \in Q$ and $t \geq 3$, then $V(C_i) \cap (\cup \mathcal{B}(x)) = \emptyset$, and if $j \in Q$ and $t = 2$, then $|V(C_j)| \geq 2|S(x) \cap V(C_j)| + 2$. If $|V(C_j)| \leq t - 1$, then $V(C_j) \cap (\cup \mathcal{B}(x)) = \emptyset$. Let $Z = \{j \notin R : V(C_j) \subseteq \cup \mathcal{B}(x) \text{ and there exists no } s^* \in S(x) \cap V(C_j) \text{ such that } |B_{s^*}| \geq t + 1 \text{ or } y \in V(C_j) \text{ such that } y \notin \cup \mathcal{B}(x)\}$, and $c = |Z|$. Note that $i \notin Z$. By arguments above, if $t \geq 3$ and $j \in Z$, then $d(j) \geq 2t - 1$ and if $t = 2$ then $d(j) \geq 1$. If $|V(C_j)| \geq t + 1$ and $j \notin Z$, then either there exists $s^* \in S(x) \cap V(C_j)$ such that $|B_{s^*}| \geq t + 1$ or there exists $y \in V(C_j)$ such that $y \notin \cup \mathcal{B}(x)$. Then

$$r \geq t|S(x)| + m - a - c + t|Q|.$$

Hence

$$|S(x)| < \frac{r - m}{t} - \frac{ta - 2a + d - c}{t}.$$

If $c > 0$, then $|S(x)| \leq \lfloor (r - m)/t \rfloor$. Assume that $c = 0$. If $ta - 2a + d \geq 0$, then $|S(x)| \leq \lfloor (r - m)/t \rfloor$. If $ta - 2a + d < 0$. Then $t < 2 - d/a$, a contradiction. Thus we assume $u \notin V(C_i)$. Let $u \in V(C_j)$, and let $D_h = C_h$ if $i \neq h \neq j$, $D_i = T_1[V(C_i) \cup \{u\}]$ and $D_j = C_j - \{u\}$. Let $Z = \{h \notin R : V(D_h) \subseteq \cup \mathcal{B}(x) \text{ and there exists no } s^* \in S(x) \cap V(D_h) \text{ such that } |B_{s^*}| \geq t + 1 \text{ or } y \in V(D_h) \text{ such that } y \notin \cup \mathcal{B}(x)\}$, and $c = |Z|$.

If $i \in Q$ and $j \in R - Q$, then $t = 2$, $|V(D_i)| = 1$, $|V(D_j)| = 1$, $V(D_i) \cap (\mathcal{B}(x)) = \emptyset$, and $|B_{s''}| \geq 3$. Hence

$$\begin{aligned}
r &\geq 2|S(x)| + m - a - c + 1 + 1 + 2(|Q| - 1) \\
&> 2|S(x)| + m - a - c + 2|Q|
\end{aligned}$$

and $|S(x)| \leq \lfloor (r - m)/2 \rfloor$.

If $i \in Q$ and $j \notin R$, then again $t = 2$, $|V(D_i)| = 1$ and $V(D_i) \cap (\cup \mathcal{B}(x)) = \emptyset$.

Then

$$\begin{aligned}
r &\geq 2|S(x)| + m - a - c - 1 + 1 + 2(|Q| - 1) \\
&> 2|S(x)| + m - 2 - c + d.
\end{aligned}$$

If $c > 0$, then $|S(x)| \leq \lfloor (r - m)/2 \rfloor$. If $c = 0$, then

$$|S(x)| < \frac{r - m}{2} - \frac{d - 2}{2}.$$

If $d \geq 1$, then $|S(x)| \leq \lfloor (r - m)/2 \rfloor$. If $d = 0$, then $d_T(w) = 2$.

As $j \notin R$, $|V(C_j)| = 1$ or $|V(C_j)| \geq 3$. Assume $|V(C_j)| \geq 3$. Then $|V(D_j)| \geq 4$.

Let $y \in V(C_j) \cap U$. then $yv' \in E(G)$. As $|V(C_j)| \geq 3$, there exists $x_1 \in V(C_j)$ such that $yx_1 \in E(T_1)$. Thus $|B_y| \geq 3$. Hence

$$r \geq 2|S(x)| + m - 1 + 1 + 2(|Q| - 1)$$

and $|S(x)| \leq \lfloor (r - m)/2 \rfloor$. We thus assume that $|V(C_j)| = 1$.

We now show that $S'(w) \leq 2$ or $|S(x)| \leq \lfloor (r - m)/2 \rfloor$. First observe that if $h \neq f$ and $V(C_h) \subseteq (\cup \mathcal{B}(v)) \cup (\cup \mathcal{B}(x))$, then w is not adjacent in G to any vertex of

$V(C_h)$. Let C_h be such that there exists $y \in V(C_h)$ such that $y \notin (\cup \mathcal{B}(v)) \cup (\cup \mathcal{B}(x))$ and $|V(C_h)| \geq 2$. Hence $h \in R$ or $|V(C_h)| \geq 3$. If $h \in R$, then $h \notin Q$. Then

$$r \geq 2|S(x)| + m - a + 2(|Q| - 1) + 2$$

and $|S(x)| \leq \lfloor (r - m)/2 \rfloor$. Hence we assume $|V(C_h)| \geq 3$. Then there exists $x_1 \in V(C_h)$ such that $yx_1 \in E(T_1)$, and as $d(h) = 0$, there exists x_2 such that $x_1x_2 \in E(T_1)$ and $x_2 \in S(v)$. As $d(h) = 0$, $d_{T_1}(x_2) = 1$. We conclude that $x_2 \notin \cup \mathcal{B}(x)$. Hence

$$r \geq 2|S(x)| + m - a + 2(|Q| - 1) + 1$$

and $|S(x)| \leq \lfloor (r - m)/2 \rfloor$. Thus if w is adjacent to some vertex of $V(C_h)$, then $|V(C_h)| = 1$ or $h = f$. Clearly w is adjacent to some vertex of at most one C_h , if $|V(C_h)| = 1$. Further, as $d(f) = 0$, w is adjacent to exactly one element of D_f . Hence $|S'(w)| \leq 2$

We now show that if $a = 1$ and $V(T_1) - U \subseteq V(D_f)$, then either the tree $T_2 = T[V(T_1) \cup \Gamma_T(v)]$ is contained in G or there exists $v'' \in V(G) - V(T_1)$ such that $wv'' \in E(G)$ and v'' is adjacent to exactly one vertex of T_1 , or $|S(x)| \leq \lfloor (r - m)/2 \rfloor$. First suppose that $|V(D_f)| > 3$. Then there exists $x_1 \in V(D_f)$ such that $x_1 \notin B_w$. As $i \in Q$ and $j \in R - Q$, $D_f = C_f$. As $d = 0$, $d(f) = 0$ and there exists $x_2 \in V(D_f)$ such that $x_1 \in B_{x_2}$, and $x_w \in S(v)$. As $d = 0$, $|B_{x_2}| = 2$ and hence $d_{T_1}(x_2) = 1$. If $x_2 \notin \cup \mathcal{B}(x)$, then

$$r \geq 2|S(x)| + m - a + 2(|Q| - 1) + 1$$

and $|S(x)| \leq \lfloor (r - m)/2 \rfloor$. Hence we assume that $x_2 \in \cup \mathcal{B}(x)$ so that $x_1 \in S(x)$.

Then $|B_{x_1}| \geq 3$ and hence $|V(D_f)| \geq 2|S(x) \cap V(D_f)| + 1$. Thus

$$r \geq 2|S(x)| + m - a + 2(|Q| - 1) + 1$$

and $|S(x)| \leq \lfloor (r - m)/2 \rfloor$. We thus assume that $|V(D_f)| = 3$ and consequently that D_f is a path of length 2. If $m \geq \lceil k/2 \rceil$, then v is adjacent to exactly two vertices of T_1 and $r = m + 1 + 2 \geq \lceil k/2 \rceil + 3$. Hence $k - r + 1 \leq \lceil k/2 \rceil - 2$, and v is adjacent to at least $\lceil k/2 \rceil - 2$ vertices not contained in $V(T_1)$. Hence T_2 is contained in G . If $m < \lceil k/2 \rceil$, then let $W = \{y : yw \in E(G), y \notin V(T_1) \text{ or } y = v\}$. We remark that the assumptions that we are currently working under will be shown to be best possible for $k = 6$ and $t = 2$. Clearly no element of W is adjacent to any element of $V(D_f)$ except w , and each element of W is adjacent to at most one element of $U - V(D_f)$. Further, we may assume that no element of W is adjacent to u . Now, $|U - V(D_f) - \{u\}| = m - 1$ and $|W| \geq \lceil k/2 \rceil - 1$. Hence $|W| > |U - V(D_f) - \{u\}|$, so there exists $v'' \in W$ so that v'' is not adjacent to any element of U . Then v'' is adjacent to exactly one vertex of T_1 . Thus we assume without loss of generality that if $a = 1$, then $V(T_1) - U \not\subseteq V(D_f)$.

If $a = 1$, then as $V(T_1) - U \not\subseteq V(D_f)$, there exists $h \notin R$ such that $|V(D_h)| \geq 3$ and $h \neq f$, $h \neq j$. Hence $k \geq |V(T_1)| \geq 9$ and so $\lceil k/2 \rceil \geq 5$. As $|S'(w)| \leq 2$, there

exists $v'' \in V(G) = V(T_1)$ such that $vv'' \in E(G)$ but $v'' \neq v'$. If $a > 1$, then as $|Q| > a/2 > 1$, $|Q| \geq 3$. Then $k \geq |V(T_1)| \geq 10$ and again $\lceil k/2 \rceil \geq 5$. As above there exists $v'' \in V(G) - V(T_1)$ such that $vv'' \in E(G)$ but $v'' \neq v'$.

If $U \subseteq \cup \mathcal{B}(v'')$ then $uv'' \in E(G)$ and G contains a 4-cycle, a contradiction. Hence $U \not\subseteq \cup \mathcal{B}(v'')$. If $u \notin \cup \mathcal{B}(v'')$ then by Case 2.2 v'' is adjacent to at most $\lfloor (r-m)/2 \rfloor$ vertices of T_1 . Hence we assume $u \in \cup \mathcal{B}(v'')$ and v'' is adjacent to some element of $U - \{u\}$.

Let $1 \leq j' \leq m$ such that v'' is adjacent to some element of $(U - \{u\}) \cap (V(C_{j'}))$. Clearly $j' \neq i$. Let $D'_h = C_h$ if $j' \neq h \neq i$, $D'_{j'} = T_1[V(C_{j'}) \cup \{u\}]$, and $D'_i = D_i = C_i - \{u\}$. If $j' \notin R$, then

$$\begin{aligned} r &\geq 2|S(v'')| + m - a - 1 + 1 + 2(a - 1) \\ &= 2|S(v'')| + m + a - 2. \end{aligned}$$

Hence

$$|S(v'')| \leq \frac{r-m}{2} - \frac{2-a}{2},$$

and if $a \geq 3$, then $|S(v'')| \leq \lfloor (r-m)/2 \rfloor$. If $a = 1$, then there exists $h \notin R$ such that $|V(C_h)| \geq 3$. Let $y \in V(C_h) \cap U$. As $d = 0$, $|V(C_h)| = 2|S(v) \cap V(C_h)| + 1$, and clearly $y \notin S(v)$. Further, by arguments above, we may assume that $|V(C_h)| = 2|S(x) \cap V(C_h)| + 1$, and we also have that $y \notin S(x)$. As $S(x) \cap S(v) = \emptyset$, if $x_1 \in V(C_h)$ such that $x_1 \neq y$, then $x_1 \in S(x) \cup S(v)$. If $h \neq j'$, then $V(C_h) \cap (\cup \mathcal{B}(v'')) = \emptyset$, and $V(C_h) = V(D'_h)$. Hence

$$r \geq 2|S(v'')| + m - 1 + 2$$

and $|S(v'')| \leq \lfloor (r - m)/2 \rfloor$. If $h = j'$, then $|B_y| \geq 4$ and

$$r \geq 2|S(v'')| + m + 2$$

and again $|S(v'')| \leq \lfloor (r - m)/2 \rfloor$.

If $j' \in R$, then $|V(D'_{j'})| = 3$ and $|V(D'_i)| = 1$. Further, if $h \in R$, $h \neq j$, then $|V(D'_h)| \geq 2|S(v'') \cap V(D'_h)| + 2$. Hence

$$r \geq 2|S(v'')| + m - a + 2 + 2(a - 2)$$

$$= 2|S(v'')| + m + a - 2.$$

As we may assume by arguments above that $j \neq i$, $a \geq 2$ and $|S(v'')| \leq \lfloor (r - m)/2 \rfloor$.

If $i \in Q$, $j \in Q$, then $V(D_i) \cap (\cup \mathcal{B}(x)) = \emptyset$, and $|B_{s''}| \geq 3$. Hence

$$r \geq 2|S(x)| + m - a - c + 1 + 1 + 2(|Q| - 2)$$

$$> 2|S(x)| + m - 2 - c + d.$$

If $c > 0$, then $|S(x)| \leq \lfloor (r - m)/2 \rfloor$. If $c = 0$, then

$$|S(x)| < \frac{r - m}{2} + \frac{2 - d}{2}.$$

If $d \geq 1$, then $|S(x)| \leq \lfloor (r - m)/2 \rfloor$. If $d = 0$ and $a = |Q| = 2$, then

$$r \geq 2|S(x)| + m - 2 + 2 + 2(2 - 2)$$

and $|S(x)| \leq \lfloor (r - m)/2 \rfloor$. If $a \geq 3$, let $w \in V(C_f)$. As $i \in Q$, $j \in Q$, $V(D_f) = V(C_f)$. As $d = 0$, $d_T(w) = 2$ and $V(D_f) \subseteq \cup \mathcal{B}(v)$.

Now, we need only find one vertex that has not already been counted. Hence we assume without loss of generality that if $h \in R - Q$, then x is adjacent in G to some element of $V(D_h)$. Further, we assume that if $h \notin R$, $h \neq f$, such that $|V(D_h)| \geq 3$, then $V(D_h) \subseteq (\cup \mathcal{B}(v)) \cup (\cup \mathcal{B}(w))$, and that $V(D_h) - U \subseteq S(v) \cup S(x)$. We conclude that w is not adjacent to any vertex of $V(D_h)$, $h \in R$, and if $h \notin R$, then w is adjacent to at most one vertex of $V(D_h)$, where $|V(D_h)| = 1$. Hence $|S'(w)| \leq 2$.

As $a \geq 3$, $|V(T_1)| \geq 9$ so that $k + 1 \geq 10$ and $\lceil k/2 \rceil \geq 5$. As $|S'(w)| \leq 2$, there exists $v'' \in V(G) - V(T_1)$, $v'' \neq v'$, such that $wv'' \in E(G)$. If $U \subseteq \cup \mathcal{B}(v'')$, then by Case 1, v'' is adjacent to at most $\lfloor (r - m)/2 \rfloor$ vertices of T_1 , and if $u \notin \cup \mathcal{B}(v'')$, then by Case 2, v'' is adjacent to at most $\lfloor (r - m)/2 \rfloor$ vertices of T_1 . Thus we assume $u \in \cup \mathcal{B}(v'')$. Let $1 \leq j' \leq m$ such that there exists $s''' \in S(v'') \cap V(C_{j'})$ and $u \in B_{s'''}'$. If $j' \notin Q$, then by replacing v' with v'' , arguments above show that either $|S(v'')| \leq \lfloor (r - m)/2 \rfloor$ or the tree $T_2 = T[V(T_1) \cup \Gamma_T(v)]$ is contained in G . Thus we assume $j' \in Q$. Let D'_1, D'_2, \dots, D'_m be as above. If $h \notin R$ and $|V(D'_h)| = 1$, then $V(D'_h) \subseteq U$ and clearly v'' is not adjacent in G to any vertex in D'_h . If $h \notin R$ and $|V(D'_h)| = 3$, $h \neq f$, then by arguments above, v or v' is adjacent in G to every vertex contained in $V(D'_h) - U$ and v'' is not adjacent in G to any vertex

in $V(D'_h) \cap U$. Let $h \in Q$, $h \neq j'$. Assume $|V(D'_h)| \geq 4$. Let $y \in U \cap V(D'_h)$. As $h \in Q$, there exists $x_1 \in V(D'_h)$ such that $y \in B_{x_1}$ and $x_1 \in S(v)$. Then $d_{T_1}(x_1) = 1$. as $|V(D'_h)| \geq 4$, there exists $x_2 \in V(D'_h)$ such that $x_2 y \in E(G)$, and $x_3 \in V(D'_h)$ such that $x_2 \in B_{x_3}$ and $x_3 \in S(v)$. Then $y_1, x_1, x_3 \notin \cup \mathcal{B}(v'')$. Hence $|V(D'_h)| \geq 2|S(v'')| + 3$ and

$$r \geq 2|S(v'')| + m - a + 2(|Q| - 2) + 3.$$

Thus $|S(v'')| \leq \lfloor (r - m)/2 \rfloor$. If $|V(D'_h)| = 2$ for all $h \neq j'$, then clearly v'' is not adjacent in G to any element of $U - V(D'_h)$, so that v'' is not adjacent in G to any vertex of $V(D'_h)$. Further, if $h \in R$, $h \neq i'$, then v or v' is adjacent in G to the vertex of $V(D'_h) - U$, so that v'' is not adjacent in G to any vertex of D'_h . Thus v'' is adjacent in G only to vertices in T_1 that are contained in $V(D'_f)$ or $V(D'_{j'})$. We conclude that $|S(v'')| = 2$. As $a \geq 3$, $r \geq m + 5$ and $\lfloor (r - m)/2 \rfloor \geq 2$. Thus $|S(v'')| \leq \lfloor (r - m)/2 \rfloor$.

If $i \in R$, then $i \in Q$, so we now consider when $i \notin R$. Then $|V(C_i)| \geq t + 1$, and $|V(D_i)| \geq t$. If $j \in R - Q$, then $|V(C_j)| = t$ so that $|V(D_j)| \geq t + 1$. Then

$$\begin{aligned} r &\geq t|S(x)| + m - a - c - 1 + 1 + t|Q| \\ &> t|S(x)| + m + a(t - 2) - c + d. \end{aligned}$$

Hence $|S(x)| \leq \lfloor (r - m)/t \rfloor$.

If $i \notin R$, $j \notin R$, then $|V(D_i)| \geq t$ and $|V(C_j)| \geq t - 1$ so $|V(D_j)| \geq t$. Then

$$\begin{aligned} r &\geq t|S(x)| + m - a - c - 2 + t|Q| \\ &> t|S(x)| + m + a(t - 2) - c + d - 2 \end{aligned}$$

If $c \geq 1$, then $|S(x)| \leq \lfloor (r - m)/t \rfloor$. If $c = 0$, then

$$|S(x)| < \frac{r - m}{t} - \frac{a(t - 2) + d - 2}{t}.$$

If $a(t - 2) - 2 + d \geq -1$, then $|S(x)| \leq \lfloor (r - m)/t \rfloor$. If $a(t - 2) - 2 + d < -1$, then

$$t < 2 + \frac{1 - d}{a}.$$

Hence $t = 2$ and $d = 0$. Thus $d_T(w) = 2$. Now if there exists one vertex of $V(T_1)$ that has not been counted, then $|S(x)| \leq \lfloor (r - m)/2 \rfloor$. Hence we assume that if $h \in R$, then $V(D_h) \subseteq (\cup \mathcal{B}(v)) \cup (\cup \mathcal{B}(x))$, $V(D_i) \subseteq \cup \mathcal{B}(x)$, $V(D_j) \subseteq \cup \mathcal{B}(x)$, and if $h \notin R$, $h \neq f$, $i \neq h \neq j$ and $|V(D_h)| \geq 3$, then $|V(D_h)| = 2|S(x) \cap V(D_h)| + 1$. Note that as $d = 0$, we also have for such h that $|V(D_h)| = 2|S(v) \cap V(D_h)| + 1$. By arguments similar to those above, we have that $V(D_h) \subseteq (\cup \mathcal{B}(v)) \cup (\cup \mathcal{B}(x))$ and if $y \in U \cap V(D_h)$, then $y \notin S(x) \cup S(v)$ and $V(D_h) - y \subseteq S(x) \cup S(v)$. By arguments analogous to those above, we conclude that $|S'(w)| = 2$. Further, $k \geq |V(T_1)| \geq 9$ so that $\lceil k/2 \rceil \geq 5$. Hence there exists $v'' \in V(G) - V(T_1)$ such that $wv'' \in E(G)$ but $v'' \neq v'$.

Observe that v or v'' is adjacent to every element of $V(T_1) - (U \cup V(C_f))$. We conclude that $|S(v'')| \leq 2$. As $a \geq 1$, and $i, j \notin R$, there are at least 4 vertices of $V(T_1)$ not contained in U so that $\lfloor (r-m)/2 \rfloor \geq 2$. Thus $|S(v'')| \leq \lfloor (r-m)/2 \rfloor$.

If $i \notin R$ and $j \in Q$, we first show that $t = 2$ or $|S(x)| \leq \lfloor (r-m)/t \rfloor$. Let $y \in V(D_j)$ such that $yv \in E(G)$. Then, as $u \notin V(C_j)$, $\text{dist}_{T_1}(u, y) = t$. Now, if $\text{dist}_{T_1}(u, s'') \leq t-2$, then $U \subseteq B_{s''}$ and by arguments in Case 1, $|S(x)| \leq \lfloor (r-m)/t \rfloor$. If $\text{dist}_{T_1}(u, s'') \geq t-1$, then $\text{dist}_{T_1}(u, s'') = t-1$ and $s''y \in E(T_1)$. Then $v'wvx s''v'$ is a 5-cycle in G , so that $t = 2$.

Now, $|B_{s''}| \geq 3$ so that

$$\begin{aligned} r &\geq 2|S(x)| + m - a - 1 + 1 + 2(|Q| - 1) \\ &> 2|S(x)| + m + d - 2 \end{aligned}$$

and

$$|S(x)| < \frac{r-m}{2} - \frac{d-2}{2}.$$

If $d > 0$, then

$$|S(x)| < \frac{r-m}{2} + \frac{1}{2}$$

so that $|S(x)| \leq \lfloor (r-m)/2 \rfloor$. If $d = 0$, $d_T(w) = d_{T_1}(w) = 2$. It follows with arguments analogous to those above that we may assume $|S'(w)| \leq 2$. Now $|T_1| \geq 8$ so that $k+1 \geq 9$ and $\lceil k/2 \rceil \geq 4$. If $\lceil k/2 \rceil = 4$, then $k = 8$ and $m = 3$. Then v'

is adjacent in G to two vertices of T_1 and hence $T \subseteq G$. If $\lceil k/2 \rceil \geq 5$, then there exists $v'' \in V(G) - V(T_1)$ such that $v''w \in E(G)$ but $v'' \neq v'$. It again follows with arguments similar to those above that v'' is adjacent to at most $\lfloor (r-m)/2 \rfloor$ vertices of T_1 .

Hence the theorem follows by induction. \square

We remark that Brandt [14] has found a significantly shorter proof of the above result.

We now show that the above result is best possible for at least one value of k if $g = 5$. In particular, we will show that if $k = 6$, then the Petersen graph does not contain every tree of maximal degree 3 and order 8. Let T be the tree in Figure 4.

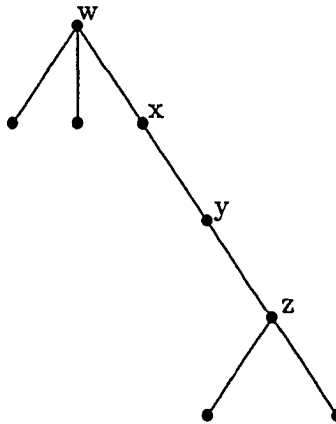


Figure 4. The tree T .

As the Petersen graph is 3-transitive [33], for any two paths of length 3, there exists an automorphism of the Petersen graph taking one to the other. Hence we may assume that if T is contained in the Petersen graph, then w, x, y , and z are the vertices labeled as in Figure 5.

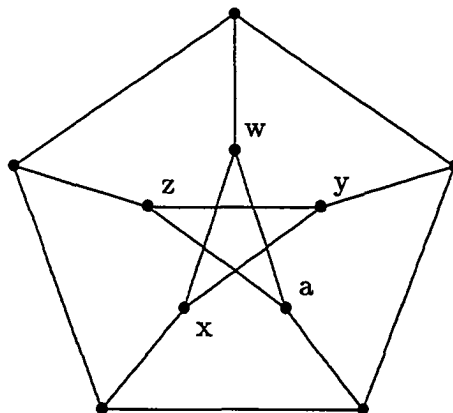


Figure 5. The Petersen graph with appropriate labeling.

Then $aw, az \in E(T)$, a contradiction. We remark that Brandt [14] has generalized the above example to show that there are at least three graphs for which Theorem 9.1 is best possible and at most four.

Note that the tree T in Figure 4 can be generalized for all $k \geq 5$ and $t \geq 2$ to show that our method will not suffice to construct all trees T of order $t\lceil k/2 \rceil + 2$ with maximal degree $\Delta(T) = \lceil k/2 \rceil$ in a graph G with girth $g = 2t + 1$ and $\delta(G) \geq \lceil k/2 \rceil$. Let T_1 be a spider such that $\Delta(T_1) = \lceil k/2 \rceil$ and $\lceil k/2 \rceil - 1$ legs have length $t - 1$, and the remaining leg has length $t + 1$. Then $|V(T_1)| = (t - 1)\lceil k/2 \rceil + 3$. Let T be the tree defined by adjoining the foot of the leg of T_1 to $\lceil k/2 \rceil - 1$ additional vertices. Then $|V(T)| = t\lceil k/2 \rceil + 2$. Assume that T_1 has been constructed in G . Let v be the foot of the leg of T_1 of length $t + 2$, and $w \in V(T_1)$ such that $wv \in E(G)$. Then v may be adjacent to a foot of a leg of length t . Further, there are $\lceil k/2 \rceil - 2$ possibilities of v' , and each of these vertices may also be adjacent in G to a foot of a leg of length t . Hence there does not necessarily exist a vertex x such that $x = v$ or

$x \in V(G) - V(T_1)$, $xw \in E(G)$, such that x is adjacent to at least $\lceil k/2 \rceil - 1$ vertices not contained in $V(T_1)$, and so we may not construct T in G using the arguments above.

Similar to Corollaries 8.4 and 8.5 we have

Theorem 9.2. Let G be a graph with girth $g \geq 5$ and $|E(G)| \geq \lfloor (k-1)n/2 \rfloor + 1$. Then G contains every tree of order $k+1$.

Corollary 9.3. If G is a graph with girth $g \geq 5$ and $|E(G)| = \lfloor (k-1)n/2 \rfloor$, then if k is odd G does not contain every tree of order $k+1$ if and only if G is $(k-1)$ -regular, and if k is even, then G does not contain every tree of order $k+1$ if and only if at most one vertex of G has degree $k-2$ and every other vertex has degree $k-1$.

Conjecture 9.4. Let G be a graph with girth $g \geq 2t+1$ and $\delta(G) \geq k \geq 3$. Then G contains every tree T of order $kt+1$ such that $\Delta(T) \leq k$.

Note that Conjecture 9.4 is trivially true for $t=1$ and that Theorem 9.1 proves that the conjecture is true for $t=2$.

Let G be a graph and let $B_t(v) = G[\{r \in V(G) : \text{dist}_G(r, v) \leq t\}]$. Then the following is simply a reformulation of the definition of girth.

Lemma 9.5. A graph G has girth $g = 2t$, $t \geq 2$, if and only if $B_t(v)$ is a tree for all $v \in V(G)$.

For a graph G and $t \geq 2$, define $r_t(v)$ to be the minimum number of edges that must be deleted from G so that $B_t(v)$ is a tree, $v \in V(G)$. Clearly $r_t(v) \leq |E(B_t(v))| - |V(B_t(v))| + 1$. Then

Corollary 9.6. Let G be a graph such that $|E(G)| \geq \lfloor (k-1)n/2 \rfloor + 1$ and

$$\sum_{v \in V(G)} r_t(v) < \frac{(k - k' - 1)n}{2}, \quad (5)$$

$t \geq 2$, $k' \geq 2k/t + (t-2)(m-1)$. Then G contains every tree of order $k+2$ such that $\delta(T) \leq \lceil k'/2 \rceil$.

Proof. If (5) holds, then there exists a spanning subgraph $G' \subseteq G$ such that $|E(G')| \geq \lfloor (k' - 1)n/2 \rfloor + 1$. Assume without loss of generality that $\delta(G') \geq \lceil k'/2 \rceil$ and

$$\ell = \frac{t}{2} - \frac{(t-2)(m-1)}{k'}.$$

As $k' \geq 2k/t + (t-2)(m-1)$, $\ell k' \geq k$ so that $\lfloor \ell k' \rfloor + 2 \geq k+2$. Thus the result follows from Theorem 9.1. □

CHAPTER 10

TREES WITH SMALL MAXIMAL DEGREE

By Corollary 9.3, if G is a graph with girth $g \geq 5$ and size $\lfloor (k-1)n/2 \rfloor$, then G contains every tree of order $k+1$ except for the k -star. This leads to the following question. If G is a graph with girth g , how many edges must G have so that G contains every tree T of order $k+1$ such that $\Delta(T) \leq m$, $m < k$? Let $f(m, \ell, g)$ be the minimum number of edges a graph G of girth g must have to contain every tree of maximal degree m and order $\ell k + 1$. By [12] Exer IV.12 and the above comments,

$$(k-1)n - \binom{k}{2} + 1 \geq f(k, k, g) \geq \lfloor (k-1)n/2 \rfloor + 1$$

for all g , and the conjecture of Erdős and Sós states that equality holds on the right.

With t, ℓ and k as in Theorem 9.1, we may restate Theorem 9.1 as

$$f(k, k, g) = \lfloor (k-1)n/2 \rfloor + 1, g \geq 5, \text{ and}$$

$$f(\lceil k/2 \rceil, \lfloor \ell k \rfloor, 2t+1) \leq \lfloor (k-1)n/2 \rfloor + 1,$$

$$t \geq 2.$$

We would like to propose the following problem:

Problem 10.1. Determine $f(k, \ell, g)$ for all k, ℓ , and g .

In this chapter, we will prove partial results on Conjectures 9.4 and 9.6, and determine some bounds $f(k, \ell, g)$.

Theorem 10.2. Let G be a graph with girth $g \geq 2t + 1$, $t \geq 3$, and $\delta(G) \geq k$. Then G contains every tree T such that $\Delta(T) \leq \lceil k/2 \rceil$ and $|V(T)| \leq kt + 1$.

Proof. We first show the result is true if T is a path. Let $S(x)$ and $\mathcal{B}(x)$ be as in Theorem 9.1. Let P be a maximal path in G , and $v \in V(P)$ such that $d_P(v) = 1$. As P is maximal, if $vv' \in E(G)$, then $v' \in V(P)$. Hence $|S(v)| = k$. Let $w \in V(P)$ such that $wv \in V(P)$. Then $|B_w| \geq t + 1$. Hence

$$|\cup \mathcal{B}(v)| \geq t(k - 1) + t + 1 = kt + 1,$$

and $T \subseteq P$. If T is not a path, then we construct T as in Theorem 9.1. Let $N(w), S(x), \mathcal{B}(x), S'(w), \mathcal{B}'(w), T_1$, and T_2 be as in Theorem 9.1. Let $S'_1(w) = \{x : x \in S'(w) \text{ but } xw \notin E(T_1)\}$, $S_2(x) = \{y \in S(x) : y \neq w\}$, $B_s^1 = \{y : \text{dist}_{T_1}(y, s) \leq t - 2\}$, $B_s^2 = \{y : \text{dist}_{T_1}(y, s) \leq t - 3\}$, $\mathcal{B}'_1(w) = \{B_s^1 : s \in S'_1(w)\}$, and $\mathcal{B}_2(x) = \{B_s^2 : s \in S_2(x)\}$. Note that $(\cup \mathcal{B}_2(x)) \cap (\cup \mathcal{B}'_1(w)) = \emptyset$ for every $x \in N(w)$ and if $v, x \in N(w)$, $x \neq v$, then $(\cup \mathcal{B}(v)) \cap (\cup \mathcal{B}'_1(w)) = \emptyset$. Assume $T_2 \not\subseteq G$. Then there exists $v \in N(w)$ such that v is adjacent in G to at least $k/2 + 1$ vertices of T_1 . Assume there does not exist $v' \in V(G) - V(T_1)$ such that $v'w \in E(G)$. Then w is adjacent in G to at least k vertices of T_1 , so that

$$|S'_1(w)| \geq k - \lceil k/2 \rceil > k/2 - 1.$$

Hence

$$\begin{aligned} r &\geq t \left(\frac{k}{2} + 1 \right) + (t-1) \left(\frac{k}{2} - 1 \right) \\ &= tk - \frac{k}{2} + 1. \end{aligned}$$

Thus $d_T(v) \leq k/2$. Assume there does not exists $x \in N(w)$ such that $x \neq v$. Then

$d_T(w) = 2$ and $|S'_1(w)| \geq k - 2$. Hence

$$tk \geq r \geq t \left(\frac{k}{2} + 1 \right) + (t-1)(k-2)$$

and $t \leq 2$, a contradiction. Hence there exists $x \in N(w)$ such that $x \neq v$.

Let $i = |N(w)|$, $N(w) = \{v_1, \dots, v_i\}$ where $d_T(v_j) \geq d_T(v_{j+1})$ and $|S(v_j)| \leq |S(v_{j+1})|$. Hence $\sum_{j=1}^i (d_T(v_j) - 1) \leq k/2$. Assume without loss of generality that $v = v_i$. Assume $d_T(v_1) \geq k/4 + 1$. Then $d_T(v_j) \leq k/4 - 1$ for all $2 \leq j \leq i$. As $T_2 \not\subseteq G$, either v_1 is adjacent in G to at least $\lceil k/2 \rceil$ vertices of T_1 or there exists $2 \leq j \leq i$ such that v_j is adjacent in G to at least $3k/4 + 1$ vertices of T_1 . If there exists v_j such that v_j is adjacent in G to at least $3k/4 + 1$ vertices of T_1 , then

$$tk \geq r \geq t \left(\frac{3k}{4} + 1 \right) + (t-1) \left(\frac{k}{2} - 1 \right)$$

and $t \leq 2$, a contradiction. Hence v_1 is adjacent in G to at least $k/2$ vertices of T_1 .

As $|S(v_i)| = |S(v)| \geq k/2 + 1$ and $v_i \neq v_1$,

$$tk \geq r \geq t\left(\frac{k}{2} + 1\right) + (t-1)\left(\frac{k}{2} - 1\right) + (t-2)\left(\frac{k}{2} - 1\right)$$

and hence $k/2 \leq 1$, a contradiction. Hence $d_T(v_1) \leq k/4$.

If $d_T(v_1) \leq k/4$, then as $T_2 \not\subseteq G$, there exists v_j such that v_j is adjacent in G to at least $3k/4$ vertices of T_1 . Hence

$$tk \geq r \geq t\frac{3k}{4} + (t-1)\left(\frac{k}{2} - 1\right)$$

and $k \leq 8$. As $k \geq 8$ and $i \geq 2$, $i \geq k/4$ and $|B_w| \geq t + k/4$. Hence

$$tk \geq r \geq t\left(\frac{3k}{4} - 1\right) + (t-1)\left(\frac{k}{2} - 1\right) + \frac{k}{4} + t$$

and $t \leq 1$, a contradiction. Hence there exists $v' \in V(G) - V(T_1)$ such that $v'w \in E(G)$.

As $T_2 \not\subseteq G$, there are at least two vertices of $N(w) \cup \{v'\}$ which are adjacent in G to at least $k/2 + 1$ vertices of T_1 . If there does not exist $v'' \in V(G) - V(T_1)$ such that $v'' \neq v'$ and $v''w \in E(G)$, then w is adjacent in G to at least $k - 1$ vertices of T_1 , and $|S'_1(w)| \geq k/2 - 2$. Hence

$$\begin{aligned} tk \geq r &\geq t\left(\frac{k}{2} + 1\right) + (t-2)\frac{k}{2} + (t-1)\left(\frac{k}{2} - 2\right) \\ &= tk + \frac{k}{2}(t-3) - t + 2 \end{aligned}$$

Thus

$$\frac{k}{2}(t-3) - t + 2 \leq 0$$

and

$$t \leq 3 + \frac{2}{k-2}.$$

As $k \geq 5$, $t = 3$. As $t = 3$, $r \geq kt - 1$ so that v or v' is adjacent in G to at least $k - 1$ vertices of T_1 . Hence

$$3k \geq 3(k-1) + \frac{k}{2} + 2\left(\frac{k}{2} - 2\right)$$

and $k \leq 14/3$, a contradiction. Hence there exists $v'' \in V(G) - V(T_1)$ such that $v'' \neq v'$ but $v''w \in E(G)$.

As $T_2 \not\subseteq G$, there are at least 3 vertices of $N(w) \cup \{v', v''\}$ that are adjacent in G to at least $k/2 + 1$ vertices of T_1 . Hence

$$tk \geq r \geq t\left(\frac{k}{2} + 1\right) + 2(t-2)\frac{k}{2}$$

and

$$t \leq \frac{4k}{k+2} < 4.$$

Hence $t = 3$. As $t = 3$, $k \geq 6$. If there does not exist $v''' \in V(G) - V(T_1)$ such that $v' \neq v''' \neq v''$ but $v'''w \in E(G)$, then $|S'_1(w)| \geq k/2 - 3$. Hence

$$3k \geq r \geq 3\left(\frac{k}{2} + 1\right) + k + 3\left(\frac{k}{2} - 3\right)$$

and $6 \leq k \leq 9$. If $k \geq 9$, then as $T_2 \not\subseteq G$, at least 4 vertices of $N(w) \cup \{v', v'', v'''\}$ are adjacent in G to at least $k/2 + 1$ vertices of T_1 and so

$$3k \geq 3\left(\frac{k}{2} + 1\right) + 3\frac{k}{2},$$

a contradiction. Hence $6 \leq k \leq 9$.

If there does not exist $x \in N(w)$ such that $x \neq v$, then $|S'_1(w)| \geq k - 4$. Hence

$$3k \geq 3\left(\frac{k}{2} + 1\right) + 2(k - 4) + 2\frac{k}{2}$$

and $k \leq 16/3$, a contradiction. As $k \leq 9$ and there exists $x \in N(w)$ such that $x \neq v$, $d_T(w) \geq 3$. Hence $|B_w| \geq 5$ and so

$$3k \geq 3\frac{k}{2} + k + 3\left(\frac{k}{2} - 3\right) + 5$$

and $k \leq 4$, a contradiction. □

Lemma 10.3. Let G be a graph with girth $g = 2t + 1$, $t \geq 4$, T a subtree of G of order r , $\Delta(T) = m$, v a vertex of T with $d_T(v) = 1$ such that $\text{dist}_T(v, u) \geq t - 1$ and

$w \in V(T)$ such that $wv \in E(T)$. Then if there exists $x \geq 1$ vertices v_1, v_2, \dots, v_x such that $wv_i \in E(G)$, $1 \leq i \leq x$, but $v_i \notin V(T)$ and v is adjacent to more than

$$\left\lfloor \frac{r-m}{(x+1)(t-2)} \right\rfloor \quad (1)$$

vertices of T , then there exists $T' \subseteq G$ such that $T' \cong T$ and if $v' \in V(T')$ such that $\text{dist}_{T'}(u, v') \geq t-1$ and $d_{T'}(v) = 1$, then v' is adjacent to at most (1) vertices of T' .

Proof. Let $u \in V(T)$ such that $d_T(u) = \Delta(T) = m$. Let $S = \{s \in V(T) : rs \in E(G), r \in \Gamma_G(w), d_T(r) = 1\} \cup \{v_i : 1 \leq i \leq x\}$, $B_s = \{r \in V(T) : \text{dist}_T(r, s) < t-2\}$, and $\mathcal{B} = \{B_s : s \in S\}$. As before the elements of \mathcal{B} are mutually disjoint and if $B_s \in \mathcal{B}$, then $|B_s| \geq t-2$, and $|B_w| \geq t-1$. We first show that $|S| \leq (r-m)/(t-2)$.

Let $U = \Gamma_T(u) \cup \{u\}$. If $U \cap B_s = \emptyset$ for all $s \in S$. Then

$$r \geq (t-2)|S| + m + 1$$

so $|S| \leq (r-m)/(t-2)$. If there exists $s \in S$ such that $U \cap B_s \neq \emptyset$, then, as $g \geq 2t+1$, s is unique. If $u \in S$, then there are $t-4$ elements of B_s that are not contained in U . Hence

$$r \geq (t-2)(|S|-1) + m + 1 + t - 4 + 1$$

and so $|S| \leq (r-m)/(t-2)$. If $U \subseteq B_s$, then as $\text{dist}_T(u, v) \geq t-1$, $s \neq w$. then there are at least 4 vertices contained on the unique uw path in T that are not

contained in $\cup \mathcal{B}$. Further, at least $t - 5$ elements of B_s are not contained in U .

Hence

$$r \geq (t - 2)(|S| - 1) + m + 1 + 4 + 1 + t - 5$$

and so $|S| \leq (r - m)/(t - 2)$. Finally, if $U \not\subseteq B_s$ but $B_s \cap U \neq \emptyset$, then there are at most two vertices of B_s contained in U and thus at least $t - 4$ elements of B_s not contained in U . Then

$$r \geq (t - 2)(|S| - 1) + m + 1 + 1 + t - 4$$

and again $|S| \leq (r - m)/(t - 2)$ as required.

Now simply observe that at most x vertices of S are adjacent to more than (1) vertices of T . □

Theorem 10.4. Let G be a graph with girth $g \geq 2t + 1$, $t \geq 4$ such that $\delta(G) \geq \lceil k/2 \rceil \geq 3$, and $1 \leq x \leq \lceil k/2 \rceil - 1$. Assume k is even and $i \equiv k/2 - m \pmod{2}$. Then G contains every tree T with $m = \Delta(T) \leq \lceil k/2 \rceil - 1$ of order $\lfloor \ell k \rfloor + 1$ where

$$\ell \leq \frac{(k - 2m + 4)^2(t - 2) + 32(m - 1) - 4i^2(t - 2)}{16k} \quad (2)$$

Proof. We construct T as in Theorem 9.1. Let T_1 be a subtree of T as in Theorem 9.1, $N(w)$ be as in Theorem 9.1, and $v \in N(w)$. By Lemma 10.3, if there exists x vertices v_1, v_2, \dots, v_x such that $wv_i \in E(G)$ but $v_i \notin V(T)$, $1 \leq i \leq x$, then there

is a tree T'_1 isomorphic to T_1 such that if v' is a vertex of T' and $v'w \in E(T')$, $d_{T'}(v') = 1$, then v' is adjacent to at most (1) vertices of T'_1 . If v is adjacent to at least $\min(\lfloor \ell k \rfloor - r + 1, m - 1) + x$ vertices not in T , then the hypothesis of Lemma 10.3 is satisfied and T'_1 exists. We first verify the inequality

$$\left\lceil \frac{k}{2} \right\rceil - \left\lfloor \frac{r - m}{(x + 1)(t - 2)} \right\rfloor \geq \min(\lfloor \ell k \rfloor - r + 1, m - 1) + x$$

is true if

$$\ell \leq \frac{(k - 2x - 2m + 2)(x + 1)(t - 2) + 4(m - 1)}{2k}. \quad (3)$$

If $\ell k - r + 1 \geq m - 1$, then

$$\begin{aligned} \ell k - r + m + (xt - 2x + t - 4)(m - 1) &= \\ \ell k - r + (xt - 2x + t - 4)(m - 1) + m - 1 + 1 &= \\ \ell k - r + (xt - 2x + t - 3)(m - 1) + 1 &\geq (x + 1)(t - 2)(m - 1). \end{aligned}$$

Hence

$$\frac{\ell k - (r - m) + (xt - 2x + t - 4)(m - 1)}{(x + 1)(t - 2)} \geq m - 1.$$

Thus we need to show that

$$\frac{\ell k + (xt - 2x + t - 4)(m - 1)}{(x + 1)(t - 2)} \leq \frac{k}{2} - x. \quad (4)$$

Now, by (3), the left hand side of (4) is greater than or equal to

$$\begin{aligned}
 & \frac{(k - 2x - 2m + 2)(x + 1)(t - 2) + 4(m - 1) + 2(xt - 2x + t - 4)(m - 1)}{2(x + 1)(t - 2)} \\
 &= \frac{k - 2x - 2m + 2}{2} + \frac{2(m - 1)}{(x + 1)(t - 2)} + \frac{(xt - 2x + t - 4)(m - 1)}{(x + 1)(t - 2)} \\
 &= \frac{k}{2} - x - m + 1 + \frac{(xt - 2x + t - 2)(m - 1)}{(x + 1)(t - 2)} \\
 &= \frac{k}{2} - x - m + 1 + \frac{(x + 1)(t - 2)(m - 1)}{(x + 1)(t - 2)} \\
 &= \frac{k}{2} - x.
 \end{aligned}$$

If $m - 1 \geq \ell k - r + 1$, then

$$(xt - 2x + t - 3)(\ell k - r + 1) + m - 1 \geq (x + 1)(t - 2)(\ell k - r + 1).$$

Hence

$$\frac{(xt - 2x + t - 3)(\ell k - r + 1) + m - 1}{(x + 1)(t - 2)} \geq \ell k - r + 1. \quad (5)$$

Now, the left hand side of (5) is equal to

$$\frac{xt - 2x + t - 4}{(x + 1)(t - 2)} + \frac{\ell k}{(x + 1)(t - 2)} - \frac{r - m}{(x + 1)(t - 2)},$$

so it suffices to show that

$$\frac{xt - 2x + t - 4}{(x + 1)(t - 2)} + \frac{\ell k}{(x + 1)(t - 2)} \leq \frac{k}{2} - x. \quad (6)$$

The left hand side of (6) is greater than or equal to

$$\begin{aligned}
 & \frac{xt - 2x + t - 4}{(x+1)(t-2)} + \frac{(k - 2x - 2m + 2)(x+1)(t-2) + 4(m-1)}{2(x+1)(t-2)} \\
 &= \frac{k}{2} - x - m + 1 + \frac{2(m-1)}{(x+1)(t-2)} + \frac{xt - 2x + t - 4}{(x+1)(t-2)} \\
 &= \frac{k}{2} - x - m + 1 + \frac{(m-1)(xt - 2x + t - 2)}{(x+1)(t-2)} \\
 &= \frac{k}{2} - x.
 \end{aligned}$$

Note that if we consider k, t , and m to be constants and require equality in (3), then ℓ is given as a function of x . An easy calculation will show that ℓ is maximized if

$$x = \frac{k - 2m}{4}.$$

Now, $m < k/2$ so that

$$m - \frac{m}{2} < \frac{k}{2} - \frac{k}{4},$$

$$m < \frac{k}{2} - \frac{k - 2m}{4} = \frac{k}{2} - x.$$

Hence

$$m \leq \left\lceil \frac{k}{2} \right\rceil - \lceil x \rceil = \left\lceil \frac{k}{2} \right\rceil - \frac{k - 2m}{4} - \frac{i}{2},$$

and (3) holds. We then have that

$$\begin{aligned}
\ell &\leq \frac{(k - 2x - 2m + 2)(x + 1)(t - 2) + 4(m - 1)}{2k} \\
&= \frac{\left(k - \frac{k-2m}{2} - i - 2m + 2\right) \left(\frac{k-2m}{4} + \frac{i}{2} + 1\right)(t - 2) + 4(m - 1)}{2k} \\
&= \frac{\left(\frac{k}{2} - i - m + 2\right) \left(\frac{k}{4} - \frac{m}{2} + \frac{i}{2} + 1\right)(t - 2) + 4(m - 1)}{2k} \\
&= \frac{(k - 2i - 2m + 4)(k + 2i - 2m + 4)(t - 2) + 64(m - 1)}{16k} \\
&= \frac{(k - 2m + 4)^2(t - 2) + 64(m - 1) - 4i^2(t - 2)}{16k}
\end{aligned}$$

and the result follows. \square

We now show that Conjecture 9.4 is true for fixed $m = \Delta(T)$ and $g \geq 9$, provided that k is sufficiently large.

Theorem 10.5. Let $\epsilon > 0$, $g \geq 9$, and $n \geq 1$. Then there exists k' such that if $k \geq k'$ then every graph G with girth g and $\delta(G) = k$ contains every tree T of order $nk + 1$ such that $\Delta(T) \leq (1 - \epsilon)k$.

Proof. Let $m = (1 - \epsilon)k/2$, and $g \geq 2t + 1$, $t \geq 4$. By Theorem 10.5, if G is a graph with girth g and $\delta(G) \geq \lceil k/2 \rceil$, then G contains every tree T of order $\lfloor \ell k \rfloor + 1$ such that $\Delta(T) \leq m$, where ℓ is given by (2). One can easily check that

$$\begin{aligned}
\lim_{k \rightarrow \infty} \ell &= \lim_{k \rightarrow \infty} \frac{(k - (1 - \epsilon)k + 4)^2(t - 2) + 32((1 - \epsilon)k/2 - 1) - 4i^2(t - 2)}{16k} \\
&= \lim_{k \rightarrow \infty} \frac{(\epsilon k + 4)^2(t - 2) + 16(1 - \epsilon)k - 32}{16k} \\
&= \lim_{k \rightarrow \infty} \frac{(\epsilon k + 4)^2(t - 2)}{16k} + 1 - \epsilon \\
&= \frac{t - 2}{16} \lim_{k \rightarrow \infty} \frac{(\epsilon k + 4)^2}{k} + 1 - \epsilon \\
&= \frac{t - 2}{16} \lim_{k \rightarrow \infty} \frac{\epsilon^2 k^2 + 8\epsilon k + 16}{k} + 1 - \epsilon \\
&= \frac{t - 2}{16} \lim_{k \rightarrow \infty} \epsilon^2 k + 7\epsilon + 1 \\
&= \infty
\end{aligned}$$

with m as above. Then the result follows with a change of variable. □

CHAPTER 11

PACKING ALMOST STARS INTO K_n

Our approach to the Tree Packing Conjecture is motivated by the following simple observation. Suppose that a sequence T_1, \dots, T_n can be packed into $K_{n+1} - v = K_n$. Let $xy \in E(T_i)$ such that $d_{T_i}(y) = 1$. Let $T'_i = (T_i - \{xy\}) \cup \{xv\}$. Then $T_1, \dots, T_{i-1}, T'_i, T_{i+1}, \dots, T_n$ can be packed into K_{n+1} and the edge xy is not an edge used in the packing. Essentially, we inductively apply the preceding observation, making necessary modifications, to find the following sufficient conditions for packing a sequence T_1, \dots, T_{n+1} into K_{n+1} given that T_1, \dots, T_n can be packed into K_n .

Lemma 11.1. Let T_1, T_2, \dots, T_n be a sequence of trees, $|V(T_i)| = i$, and $x_i \in V(T_i)$. Let $E_i = \{\vec{x_i y} : x_i y \in E(T_i) \text{ and } d_{T_i}(y) = 1\}$. Define a digraph D by $V(D) = [n]$ and $E(D) \subseteq \cup_{i=1}^n E_i$. Let T_{n+1} be a tree of order $n+1$, $v \in V(T_{n+1})$ such that $d_{T_{n+1}}(v) = \Delta(T_{n+1})$. Define a digraph $\vec{T_{n+1}}$ by $V(\vec{T_{n+1}}) = V(T_{n+1})$ and $E(\vec{T_{n+1}}) = \{\vec{xy} : xy \in E(T_{n+1}) \text{ and } \text{dist}_{T_{n+1}}(x, v) > \text{dist}_{T_{n+1}}(y, v)\}$. If T_1, \dots, T_n can be packed into K_n and $\vec{T_{n+1}} - v \subseteq D$, then T_1, \dots, T_n, T_{n+1} can be packed into K_{n+1} .

Proof. Let $v \in K_{n+1}$ and P_0 a packing of T_1, \dots, T_n into $K_{n+1} - v = K_n$. We will show that there exists a packing P' of T_1, \dots, T_n into K_{n+1} such that if $\vec{xy} \in \vec{T_{n+1}} - v \subseteq D$, then $xy \notin E(P')$, and that if $x \in V(\vec{T_{n+1}} - v)$, $d_{T_{n+1}}(x) > 1$, and

$xv \in E(T_{n+1})$, then $xv \notin E(P')$. Then $T_{n+1} \subseteq K_{n+1} - E(P')$ and $P = P' \cup (K_{n+1} - E(P'))$ is a packing of T_1, \dots, T_{n+1} into K_{n+1} .

We proceed by induction on $|E(\vec{T}_{n+1} - v)|$. If $|E(\vec{T}_{n+1} - v)| = 1$, and $\vec{xy} \in E(\vec{T}_{n+1} - v)$, then there exists $1 \leq i \leq n$ such that $x = x_i$, $x_i y \in E(T_i)$, and $d_{T_i}(y) = 1$. Further, $\text{dist}_{T_{n+1}}(v, x_i) > \text{dist}_{T_{n+1}}(v, y)$ so that $vy \in E(T_{n+1})$, $vx \notin E(T_{n+1})$. Let $T'_i = (T_i - x_i y) \cup \{vx\}$. Then $T'_i \cong T_i$. Let

$$P' = T_1, T_2, \dots, T_{i-1}, T'_i, T_{i+1}, \dots, T_n.$$

Then $x_i y \notin E(P')$ and if $x' \in V(\vec{T}_{n+1} - v)$, $d_{T_{n+1}}(x') > 1$ and $x'v \in E(T_{n+1})$, then $x' = y$ and $vy \notin E(P')$.

Let $k \geq 1$, and T_{n+1} a tree of order $n+1$ such that $|E(\vec{T}_{n+1} - v)| = k+1$. Let $\vec{xy} \in E(\vec{T}_{n+1} - v)$ such that $d_{T_{n+1}}(x) = 1$. Let $T_{n+1}^* = (T_{n+1} - xy) \cup \{vx\}$. Then $|E(T_{n+1}^* - v)| = k$, and so by the induction hypothesis there exists a packing P_k of T_1, \dots, T_n into K_{n+1} such that if $\vec{ab} \in \vec{T}_{n+1}^* - v$, then $ab \notin E(P_k)$ and if $a \in V(\vec{T}_{n+1}^* - v)$, $d_{T_{n+1}^*}(a) > 1$ and $av \in E(T_{n+1}^*)$, then $av \notin E(P_k)$. Now, $\vec{T}_{n+1} - v \subseteq D$ so that $\vec{T}_{n+1}^* - v \subseteq D$ and $\vec{xy} \in E(D)$. As $\vec{xy} \in E(D)$, $x = x_i$ for some i , $x_i y \in E(T_i)$, and $d_{T_i}(y) = 1$. Let $T'_i = (T_i - x_i y) \cup \{x_i v\}$. Note that $d_{T_{n+1}^*}(x) = 1$, $d_{T_{n+1}^* - v}(x) = 0$ so that $v \notin V(T_i)$. Hence T'_i is a tree and clearly $T_i \cong T'_i$. Hence $P' = T_1, T_2, \dots, T_{i-1}, T'_i, T_{i+1}, \dots, T_n$ is a packing of T_1, \dots, T_n into K_{n+1} . By induction, if $\vec{ab} \in T_{n+1} - v$ such that $\vec{ab} \neq \vec{xy}$, then $ab \notin E(P')$. Further, if $a \in V(\vec{T}_{n+1} - v)$ such that $d_{T_{n+1}}(a) > 1$ and $av \in E(T_{n+1})$, then $av \notin E(P')$. By

definition of T'_i , $xy \notin E(P')$. If $d_{T_{n+1}^*}(y) > 1$ and $vy \in E(T_{n+1}^*)$, then $vy \notin E(P')$. If $d_{T_{n+1}^* - v}(y) = 1$, then $\{xy\}$ is a component of $T_{n+1} - v$. As $d_{T_{n+1}}(x) = 1$, $yv \in E(T_{n+1})$, so that $yv \in E(T_{n+1})$ and $yv \in E(T_{n+1}^*)$. Hence $yv \notin E(T_j)$, $j \neq i$, by induction, and, by induction and construction, $yv \notin V(T'_i)$. Thus $yv \notin E(P')$. Hence the result follows by induction. \square

For a graph G , let $n_1(G)$ be the number of vertices of G with degree at least 1. We now prove the main result of this chapter.

Theorem 11.2. Let T_1, \dots, T_n be a sequence of trees, $|V(T_i)| = i$, such that for each i there exists $x_i \in V(T_i)$ such that $n_1(T_i - x_i) \leq \sqrt{6(i-1)}/4$. Then T_1, \dots, T_n can be packed into K_n .

Proof. We proceed by induction on n . If $n < 12$, then $n_1(T_i - x_i) = 0$ for all $1 \leq i \leq n$, and so each T_i is a star of order i . It is then easy to show that T_1, \dots, T_n can be packed into K_n . Let $n \geq 12$ and T_1, \dots, T_n be as above. Inductively assume that if $V(K_n) = [n]$, then $x_i = i$ and T_1, \dots, T_n can be packed into K_n . Let T_{n+1} be a tree such that there exists $x_{n+1} \in V(T_{n+1})$ and $n_1(T_{n+1} - x_{n+1}) \leq \sqrt{6n}/4$. Define a digraph D by $V(D) = [n]$ and $E(D) = \{\vec{ij} : ij \in T_i, \text{ and } d_{T_i}(j) = 1\}$. By Lemma 11.1, T_1, \dots, T_{n+1} can be packed into K_{n+1} if $\vec{T}_{n+1} - x_{n+1} \subseteq D$. Note that if G is a transitive tournament of order m , then G contains every tree \vec{T} , where \vec{T} is a rooted tree of order m with every edge directed toward the root.

As $n_1(T_{n+1} - x_{n+1}) \leq \sqrt{6n}/4$, it suffices to show that D contains a transitive tournament of order $\sqrt{6n}/4$.

As $x_i = i$, $d_D^+(i) \geq i - \sqrt{6i}/4$. Further, if $\vec{ij} \in D$ and $j > i$, then by induction $ij \in E(T_k - x_k)$ for some $k > i$. Define a digraph D' by $V(D') = [n]$ and $E(D') = \{\vec{ij} : \vec{ij} \in E(D) \text{ and } j < i\}$. Let G be the underlying simple graph of D' . It then suffices to show that G contains a clique of order at least $\sqrt{6n}/4$. Now,

$$\begin{aligned} |E(G)| &\geq \binom{n}{2} - 2 \sum_{i=1}^n \frac{\sqrt{6(i-1)}}{4} \\ &\geq \binom{n}{2} - \frac{\sqrt{6}}{2} \int_0^n \sqrt{id}i \\ &\geq \binom{n}{2} - \frac{\sqrt{6}}{3} n^{3/2} \end{aligned}$$

and so $|E(\tilde{G})| \leq \sqrt{6}/3(n+1)^{3/2}$, where \tilde{G} is the complement of G . By [13], G contains a clique of size t , where $t^2 \leq n$ and $|E(\tilde{G})| \leq n^2/2t$. It is then easy to verify that G contains a clique of order $\sqrt{6n}/4$. \square

Note that it is easy to show that Lemma 11.1 implies that if T_n is a tree of order n , then $S_1, S_2, \dots, S_{n-1}, T_n$ can be packed into K_n , where S_i is a star of order i , and Gyárfás and Lehel [32] have shown that if T_1, \dots, T_n are a sequence of trees with at most two T_i 's not stars, then T_1, \dots, T_n can be packed into K_n . We now show that there are many more sequences T_1, \dots, T_n with T_n an arbitrary tree of order n that can be packed into K_n .

Theorem 11.3. Let $c = (\sqrt{15} - 3)/3$ and T_n be a tree of order n . Let T_1, \dots, T_{n-1} be a sequence of trees, $|V(T_i)| = i$, such that for each $1 \leq i \leq n-1$ there exists $x_i \in V(T_i)$ such that $n_1(T_i - x_i) \leq c\sqrt{i-1}$. Then T_1, \dots, T_n can be packed into K_n .

Proof. As T_n is a tree, there exists $u_{n-1} \in V(T_n)$ such that $d_T(u_{n-1}) = 1$. Let $e_{n-1} \in E(T_n)$ such that u_{n-1} is incident to e_{n-1} . Let $T_n^1 = T_n - u_{n-1}$. Then T_n^1 is a tree and hence there exists $u_{n-2} \in V(T_n^1)$ such that $d_{T_n^1}(u_{n-2}) = 1$. Let $e_{n-2} \in E(T_n^1)$ such that u_{n-2} is incident to e_{n-2} . Let $T_n^2 = T_n^1 - u_{n-2}$. Continuing inductively, we get vertices $u_{n-1}, u_{n-2}, \dots, u_1 \in V(T_n)$, edges $e_{n-1}, \dots, e_1 \in E(T_n)$, and trees $T_n^1, T_n^2, \dots, T_n^{n-1}$ such that $T_n^1 = T_n - u_{n-1}$, $T_n^{i+1} = T_n^i - u_{n-i}$, and $d_{T_n^i}(u_{n-i}) = 1$. Let $u_n \in V(T_n^{n-1})$. Without loss of generality, assume $u_{n-i} = i$. We will show that there exists a packing P of T_1, \dots, T_{n-1} into K_n such that $K_n - \cup_{i=1}^{n-1} E(T_i) = T_n$.

First embed T_n in to K_n . We proceed by induction on $1 \leq i \leq n-1$. If $i = 1$ or 2 , the result follows by choice of u_1, u_2 . Let $i \geq 2$ and assume there is a packing of T_1, T_2, \dots, T_i into K_{i+1} such that $x_j = j$ for every $1 \leq j \leq i$, and $T_n[i+1] \subseteq K_{i+1} - \cup_{j=1}^i E(T_j)$. By choice of u_{n-i-2} , $d_{T_n^{i+2}}(i+2) = 1$. Hence $i+2$ is adjacent in T_n to exactly one vertex of $[i+1]$, say t . Define a digraph D' by $V(D') = [i+1]$ and $E(D') = \{\vec{k\ell} : k\ell \in E(T_k), \ell < k, \text{ and } d_{T_k}(\ell) = 1\}$. Let $D = D' - t$. By Lemma 11.1, T_1, \dots, T_{i+1} can be packed into K_{i+2} such that $T_n[i+2] \subseteq K_{i+2} \cup_{j=1}^{i+1} E(T_j)$ if D contains a transitive tournament of order $c\sqrt{i}$ whose underlying simple graph is edge disjoint from $E(T_n[i+1])$. Let G be the

underlying simple graph of D . It then suffices to show that G contains a clique of size $c\sqrt{i}$ that is edge disjoint from $E(T_n[i+1])$. Now, $E(T_n[i+1]) = i$. Let $G^* = E(T_n[i+1])$. Then

$$\begin{aligned} |E(G^*)| &\geq \binom{i}{2} - 2c \sum_{j=1}^i \sqrt{j-1} - 2i \\ &\geq \binom{i}{2} - 2c \int_0^i \sqrt{j} dj - 2i \\ &\geq \binom{i}{2} - \frac{4c}{3} i^{3/2} - 2i. \end{aligned}$$

Hence

$$\begin{aligned} |E(\tilde{G}^*)| &\leq \frac{4ci^{3/2}}{3} + 2i \\ &\leq \frac{4ci^{3/2}}{3} + 2i^{3/2} \\ &= \frac{4c+6}{3} i^{3/2}. \end{aligned}$$

Hence by [13] G^* contains a clique of order at least $c\sqrt{i}$. □

We would like to make some observations concerning both of the above theorems. First, the method which we use to find the tree $\overrightarrow{T_n} - v$ in D is a weak form of Turan's Theorem, and is most likely not a very efficient way of doing so. Second, we do not actually use all of the edges in D , which makes it very likely that both results could be improved, although the improvement would only be in the constant unless a more sophisticated method of constructing trees in D is used. Finally, it seems likely that with additional requirements on T_n (perhaps restricting the diameter of T_n) that the bound on $n_1(T_i - v)$ could be improved to a linear bound.

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VITA

The author was born in Athens, Georgia on August 20, 1965. He attended High School at Northwest High School in Justin, Texas and graduated in 1983. He received the Bachelor of Science degree in Mathematics at The University of North Texas in 1988. He subsequently received the Master of Arts degree in Mathematics at The University of North Texas 1989. He then earned the Doctor of Philosophy degree in Mathematics in 1995.

DOCTORAL EXAMINATION AND DISSERTATION REPORT

Candidate: Edward Tauscher Dobson

Major Field: Mathematics

Title of Dissertation: Some Problems in Algebraic and Extremal Graph Theory

Approved:

Reha Bollbolis

Major Professor and Chairman

Stanley L. Gerber

Dean of the Graduate School

EXAMINING COMMITTEE:

Bagdan Gromov

[Signature]

James R. Roberts

J. Hurrelbrink

P. Hilmer

James G. Oxley

Date of Examination:

December 12, 1994